Problem 1: Let $p$ be an odd prime and consider the group $D_p$.

a. Write out (with justification) the class equation of $D_p$.

b. Find all normal subgroups of $D_p$.

Problem 2: Let $G$ be a finite group. Prove that $G$ is solvable if and only if there exists a chain of subgroups

$$\{e\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H_n = G$$

such that each $H_i$ is a normal subgroup of $H_{i+1}$ and each $(H_{i+1} : H_i)$ is prime.

Problem 3: Let $G$ be a group. Given $a, b \in G$ we define the commutator of $a$ and $b$ to be $[a, b] = a^{-1}b^{-1}ab$.

Let $G'$ be the subgroup of $G$ generated by all of the commutators, i.e.

$$G' = \langle \{[a, b] : a, b \in G\} \rangle$$

a. Show that the inverse of a commutator is a commutator.

b. Show that a conjugate of a commutator is a commutator.

c. Show that $G'$ is a normal subgroup of $G$.

d. Let $N$ be a normal subgroup of $G$. Show that $G/N$ is abelian if and only if $G' \subseteq N$. Thus, $G/G'$ is the “largest” abelian quotient of $G$.

e. Define a sequence $G(n)$ recursively by letting $G(0) = G$ and $G(n+1) = (G(n))'$. Thus, $G(1) = G'$, $G(2) = G''$, etc. Show that $G$ is solvable if and only if there exists $n \in \mathbb{N}$ with $G(n) = \{e\}$.

Problem 4: Let $f(x) \in \mathbb{Q}[x]$. Working in $\mathbb{C}$, let $F$ be the splitting field of $f(x)$ over $\mathbb{Q}$. Suppose that $[F : \mathbb{Q}]$ is odd. Show that every root of $f(x)$ in $\mathbb{C}$ is real.

Problem 5: Let $F$ be a finite field with $|F| = p^n$ and consider the Galois extension $\mathbb{Z}/p\mathbb{Z} \prec F$. Let $N : F \rightarrow \mathbb{Z}/p\mathbb{Z}$ be the norm of the extension $\mathbb{Z}/p\mathbb{Z} \prec F$ as defined in Homework 7. Let

$$d = \frac{p^n - 1}{p - 1}$$

a. Let $\sigma : F \rightarrow F$ be the Frobenius automorphism, i.e. $\sigma(a) = a^p$. Show that $Gal_{\mathbb{Z}/p\mathbb{Z}}F = \langle \sigma \rangle$.

b. Show that $N(a) = a^d$ for all $a \in F$.

c. Given a field $K$, let $K^\times = K \setminus \{0\}$ considered as a multiplicative group. Notice that $N(a) \neq 0$ for all $a \neq 0$.

Letting $\varphi$ be the restriction of $N$ to $F^\times$, we know from Homework 7 that $\varphi : F^\times \rightarrow \mathbb{Z}/p\mathbb{Z}^\times$ is a group homomorphism (this is also immediate in this case from the formula in part b).

Show that $|\ker(\varphi)| = d$ and $|\text{range}(\varphi)| = p - 1$.

d. Show that $N$ is surjective.