Homework 5 : Due Wednesday, March 2

Problem 1: Prove that $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ is a normal extension of $\mathbb{Q}$.

Problem 2: Working in $\mathbb{C}$, find the splitting field over $\mathbb{Q}$ of each of the following and compute its degree.

a. $x^4 - 2$

b. $x^4 + x^2 + 1$

c. $x^6 - 4$

Problem 3: Let $p(x) = x^3 - 3x^2 + 6x + 1 \in \mathbb{Q}[x]$.

a. Show that $p(x)$ has a unique real root.

b. Working in $\mathbb{C}$, show that the splitting field of $p(x)$ over $\mathbb{Q}$ has degree 6.

Problem 4: Suppose that $\mathbb{Q} \triangleleft F \triangleleft \mathbb{C}$ and $[F : \mathbb{Q}] = 2$.

a. Show that there exists $r \in \mathbb{Q}$ with $F = \mathbb{Q}(\sqrt{r})$.

b. Let $a, b \in \mathbb{Z}$ with $b > 0$. Show that $\mathbb{Q}(\sqrt{\frac{a}{b}}) = \mathbb{Q}(\sqrt{ab})$.

c. A nonzero integer $d$ is squarefree if it is not divisible by $p^2$ for any prime $p$. Show that there exists a squarefree $d \in \mathbb{Z}$ with $F = \mathbb{Q}(\sqrt{d})$.

Problem 5: Suppose that $K \triangleleft F$ is a finite extension. Let $g(x) \in K[x]$ be irreducible and suppose that $\deg(g(x)) = p$ a prime. Show that if $g(x)$ is not irreducible in $F[x]$, then $p \mid [F : K]$.

Hint: First extend $F$ to a field $L$ where $g(x)$ has a root.