Homework 4 : Due Wednesday, February 23

Problem 1: Let $K \preceq F$ and $L \preceq E$ be finite extensions. Let $\alpha : K \to L$ be an isomorphism and let $\tau : F \to E$ be an isomorphism which is an extension of $\alpha$, i.e. $\tau(a) = \alpha(a)$ for all $a \in K$. Show that $[F : K] = [E : L]$.

Problem 2:

a. Show that $\sqrt{3} / \in \mathbb{Q}(\sqrt{2})$.

b. Show that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

c. Show that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

d. Find a fourth degree monic polynomial $p(x) \in \mathbb{Q}[x]$ that has $\sqrt{2} + \sqrt{3}$ as a root.

e. Use parts b, c, and d to conclude that $p(x)$ is the minimal polynomial of $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

Problem 3: Suppose that $K \preceq F$. Let $u \in F$ be algebraic over $K$.

a. Give an example of the above situation where $K(u) \neq K(u^2)$.

b. Suppose that the minimal polynomial of $u$ over $K$ has odd degree. Show that $K(u) = K(u^2)$.

Problem 4:

a. Show that $\mathbb{Q}(e^{2\pi i/3}) = \mathbb{Q}(i\sqrt{3})$.

b. Find $[\mathbb{Q}(e^{2\pi i/11}, \sqrt[7]{7}) : \mathbb{Q}]$.

c. Find $[\mathbb{Q}(31 + 7\sqrt{2} - 13\sqrt{8} + 42\sqrt{16}) : \mathbb{Q}]$.

Problem 5: Let $a_1, a_2, \ldots, a_n \in \mathbb{Q}$ with each $a_i > 0$. Show that $\sqrt[3]{2} \notin \mathbb{Q}(\sqrt[3]{a_1}, \sqrt[3]{a_2}, \ldots, \sqrt[3]{a_n})$.

Problem 6: Let $K = \mathbb{Z}/2\mathbb{Z}$ and let $p(x) = x^3 + x + 1 \in K[x]$. Notice that $p(x)$ is irreducible in $K[x]$ because it has degree 3 and has no roots in $K$. In class, we showed how to construct a extension of $K$ in which $p(x)$ has a root by considering the field $F = K[x]/(x^3 + x + 1)$.

If we let $u = \overline{x}$, then we can write

$$F = \{a + bu + cu^2 : a, b, c \in \mathbb{Z}/2\mathbb{Z}\}$$

where we add in the obvious way and multiply using the fact that $u^3 + u + 1 = 0$ and hence $u^3 = -u - 1 = u + 1$.

a. Write out an $8 \times 8$ tables giving addition and multiplication in $F$.

b. Factor the polynomial $x^3 + x + 1$ into irreducibles in $F[x]$.

c. Find the minimal polynomial of $u + 1 \in F$ over $K$. 