Homework 3 : Due Wednesday, February 16

Problem 1: Let $V$ and $W$ be vector spaces over a field $F$ and assume that $\dim_F V < \infty$. Let $T : V \to W$ be a linear transformation (i.e. $T(u + v) = T(u) + T(v)$ and $T(\lambda u) = \lambda T(u)$ for all $u, v \in V$ and $\lambda \in F$).

a. Show that $\ker(T)$ is a subspace of $V$ and $\text{range}(T)$ is a subspace of $W$.

b. Show that $\dim_F V = \dim_F \ker(T) + \dim_F \text{range}(T)$.

Problem 2: Let $K$ and $F$ be fields with $K \prec F$.

a. Show that $K = F$ if and only if $\left[ F : K \right] = 1$.

b. Suppose that $f(x), g(x) \in K[x]$ and that $f(x) \mid g(x)$ in $F[x]$. Show that $f(x) \mid g(x)$ in $K[x]$.

Problem 3: Although we have not worked through all of the details, the fundamental results of linear algebra work over any field. In particular, given an $n \times n$ matrix $A$ over a field $F$, the following are equivalent:

- $A$ is invertible.
- The columns of $A$ are linearly independent in $F^n$.
- The columns of $A$ span $F^n$.

Use this to determine the number of invertible $n \times n$ matrices over $F = \mathbb{Z}/p\mathbb{Z}$ for a prime $p$.

Problem 4: Find the minimal polynomial of each of the following over $\mathbb{Q}$.

a. $\sqrt{2} + 1$

b. $\sqrt{2} + i$

c. $e^{2\pi i/8} = \cos(\pi/4) + i \sin(\pi/4) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

Problem 5: Let $f(x) = x^4 + 6x - 2 \in \mathbb{Q}[x]$. Let $u$ be some (any) root of $f(x)$ in $\mathbb{C}$.

a. Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$.

b. We know that $\mathbb{Q}(u) = \{ a + bu + cu^2 + du^3 : a, b, c, d \in \mathbb{Q} \}$. Find values of $a, b, c, d$ for $u^6 - 2u^3$ and $1/u$.

Problem 6: Suppose that $F \subseteq R$ where $R$ is an integral domain and $F$ is a field which is a subring of $R$.

a. Give an example of the above situation where $R$ is not a field.

b. In this situation, we can still view $R$ as a vector space over $F$. Show that if $\dim_F R < \infty$, then $R$ is a field.

Hint: Fix $a \in R$ with $a \neq 0$. To find an inverse for $a$, define a certain linear transformation on $R$ using $a$ and think about the range.