Problem 1: Find the minimal polynomial of each of the following over \( \mathbb{Q} \).
   a. \( 3\sqrt{2} + 1 \)
   b. \( \sqrt{2} - \sqrt{2} \)
   c. \( \sqrt{3} - 2\sqrt{2} \).

Problem 2:
   a. Show that \( x^5 + x^2 + 1 \in (\mathbb{Z}/2\mathbb{Z})[x] \) is irreducible in \( (\mathbb{Z}/2\mathbb{Z})[x] \).
   b. Show that \( 3x^5 + 10x^4 - x^2 + 5 \) is irreducible in \( \mathbb{Q}[x] \).

Problem 3: Let \( p(x) = x^3 + 9x + 6 \in \mathbb{Q}[x] \).
   a. Show that \( p(x) \) is an irreducible polynomial in \( \mathbb{Q}[x] \) with a unique real root.
   b. Let \( \alpha \) be any root of \( p(x) \). We know that
      \[
      \mathbb{Q}(\alpha) = \{ a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q} \}
      \]
      Find the multiplicative inverse of \( 1 + \alpha \in \mathbb{Q}(\alpha) \) and write it in the form \( a + b\alpha + c\alpha^2 \) where \( a, b, c \in \mathbb{Q} \).

Problem 4: Let \( \pi \in \mathbb{Z}[i] \) be prime and let \( I = \langle \pi \rangle \). Show that \( \alpha^{N(\pi)} + I = \alpha + I \) for all \( \alpha \in \mathbb{Z}[i] \).
   Hint: This should resemble an important fact about \( \mathbb{Z} \).

Problem 5: Let \( R \) be a UFD with finitely many units. Show that every nonzero element \( r \) has finitely many divisors in \( R \), and give a formula for the number of such divisors based on a factorization of \( r \) into irreducibles.

Problem 6: Determine all \( (x, y) \in \mathbb{Z}^2 \) satisfying \( x^3 = y^2 + 4 \).
   Hint: Break this up into cases based on whether \( y \) is even or odd. When \( y \) is even, make use of Problem 5 on Homework 6.