Problem 1: Determine, with proof, whether the following pairs of groups are isomorphic.
   a. $A_6$ and $S_5$.
   b. $\mathbb{Z}/84\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$.
   c. $U(\mathbb{Z}/18\mathbb{Z})$ and $\mathbb{Z}/6\mathbb{Z}$.
   d. $S_4$ and $\mathbb{Z}/6\mathbb{Z} \times U(\mathbb{Z}/5\mathbb{Z})$.
   e. $A_4$ and $D_6$.
   f. $U(\mathbb{Z}/15\mathbb{Z})$ and $U(\mathbb{Z}/10\mathbb{Z})$.
   g. $S_3 \times \mathbb{Z}/2\mathbb{Z}$ and $A_4$.
   h. $D_4/\langle D_4 \rangle$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Hint: Don’t try to build explicit isomorphisms or rule out each possibility. Use the theory we have developed.

Problem 2: Consider the group $G = U(\mathbb{Z}/15\mathbb{Z})$. Find nontrivial cyclic subgroups $H$ and $K$ of $G$ such that $G$ is the internal direct product of $H$ and $K$. Use this to find $m, n \in \mathbb{N}$ with $m, n \geq 2$ such that $U(\mathbb{Z}/15\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Problem 3: Let $G$ be a group and let $H$ be a normal subgroup of $G$.
   a. Show that $G/H$ is abelian if and only if $a^{-1}b^{-1}ab \in H$ for all $a, b \in G$.
   b. Suppose $[G : H]$ is finite and let $m = [G : H]$. Show that $a^m \in H$ for all $a \in G$.

Problem 4:
   a. Let $G$ be a group and let $H$ be a subgroup of $G$. Let $g \in G$. Show that the set
      \[ gHg^{-1} = \{ ghg^{-1} : h \in H \} \]
      is a subgroup of $G$ and that $|gHg^{-1}| = |H|$ (where $|A|$ means the number of elements in the set $A$).
   b. Let $G$ be a group. Suppose that $k \in \mathbb{N}^+$ is such that $G$ has a unique subgroup of order $k$. If $H$ is the unique subgroup of $G$ of order $k$, show that $H$ is a normal subgroup of $G$. 