Homework 6 : Due Wednesday, February 19

Problem 1: Let $n \geq 3$. Show that every element of $A_n$ can be written as a product of 3-cycles (so the set of 3-cycles generates $A_n$).

Problem 2: Suppose that $\sigma \in A_n$ and $|\sigma| = 2$. Show that there exists $\tau \in S_n$ with $|\tau| = 4$ and $\tau^2 = \sigma$.

Problem 3: This problem gives another interpretation of $D_n$ as a subgroup of $GL_2(\mathbb{R})$ by thinking of rotation and flips as linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.
   a. Let $\alpha, \beta \in \mathbb{R}$. Show that
      \[
      \begin{pmatrix}
      \cos \alpha & -\sin \alpha \\
      \sin \alpha & \cos \alpha
      \end{pmatrix}
      \begin{pmatrix}
      \cos \beta & -\sin \beta \\
      \sin \beta & \cos \beta
      \end{pmatrix}
      =
      \begin{pmatrix}
      \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\
      \sin(\alpha + \beta) & \cos(\alpha + \beta)
      \end{pmatrix}
      \]
   b. Let $n \geq 3$. Let
      
      \[
      R = \begin{pmatrix}
      \cos(2\pi/n) & -\sin(2\pi/n) \\
      \sin(2\pi/n) & \cos(2\pi/n)
      \end{pmatrix}
      \quad
      S = \begin{pmatrix}
      -1 & 0 \\
      0 & 1
      \end{pmatrix}
      \]
      
      Show that $|R| = n$, $|S| = 2$, and $SR = R^{-1}S$.

Problem 4: Let $n \geq 3$. Working in $D_n$, determine $|r^k s^\ell|$ for each $k, \ell \in \mathbb{N}$ with $0 \leq k \leq n-1$ and $0 \leq \ell \leq 1$.

Problem 5: Let $n \geq 3$.
   a. Show that if $a \in D_n$ and $a \in \langle r \rangle$, then $sa = a^{-1}s$.
   b. Show that if $a \in D_n$ but $a \notin \langle r \rangle$, then $ra = ar^{-1}$.
   c. Find $Z(D_n)$. You answer will depend on whether $n$ is even or odd.

Problem 6: Compute the left cosets of the subgroup $H$ of the given group $G$ in each of the following cases (make sure you completely determine $H$ first!).
   a. $G = U(\mathbb{Z}/18\mathbb{Z})$ and $H = \langle 17 \rangle$ (you computed the Cayley table of $U(\mathbb{Z}/18\mathbb{Z})$ in Homework 4).
   b. $G = D_4$ and $H = \langle r^2 s \rangle$.
   Hint: Save as much work as you can by using the general fact that you are working with equivalence classes of a certain equivalence relation, and you know that the equivalence classes partition $G$. 

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