Homework 15: Due Friday, May 2

**Problem 1:** Recall that a Boolean ring is a ring $R$ for which $a^2 = a$ for all $a \in R$. In Homework 12, we proved every Boolean ring is commutative.

a. Show that if $R$ is both a Boolean ring and an integral domain, then $R \cong \mathbb{Z}/2\mathbb{Z}$.
b. Show that if $R$ is a Boolean ring and $I$ is an ideal of $R$, then $R/I$ is a Boolean ring.
c. Show that every prime ideal in a Boolean ring is a maximal ideal.

**Problem 2:** Let $R$ be an integral domain.

a. Show that every associate of an irreducible element of $R$ is irreducible.
b. Show that every associate of a prime element of $R$ is prime.

**Problem 3:**

a. Find, with proof, all irreducible polynomials in $\mathbb{Z}/2\mathbb{Z}[x]$ of degree 2 or 3.
b. Show that $x^2 + x^2 + 1 \in \mathbb{Z}/2\mathbb{Z}[x]$ is irreducible.

**Problem 4:** Determine whether the following polynomials are irreducible in $\mathbb{Q}[x]$.

a. $x^4 - 5x^3 + 3x - 2$
b. $x^4 - 2x^3 + 2x^2 + x + 4$

**Problem 5:** Let $R$ be an integral domain and let $a, c, d \in R$. Show that if $c$ and $d$ are associates in $R$, then $\text{ord}_c(a) = \text{ord}_d(a)$.

**Problem 6:** Show that if $R$ is a UFD, then every irreducible element of $R$ is prime.