Problem 1: Compute, with explanation, the conjugacy classes and the Class Equation for $D_5$.

Note: You can really cut down on computations using the ideas from class. It is even possible to compute the Class Equation first and use it to do very few computations.

Problem 2: Determine which finite groups that have exactly two conjugacy classes.

Problem 3: Suppose that $G$ is a nonabelian group with $|G| = 125$. Show that $|Z(G)| = 5$ and that $G/Z(G) \cong \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Problem 4: Let $n, p \in \mathbb{N}^+$ and assume that $p$ is prime. Let

$$X = \{(1, 2, \ldots, n)^p = \{(a_1, a_2, \ldots, a_{p-1}, a_p) : 1 \leq a_i \leq n \text{ for all } i\}$$

be the set of all $p$-tuples such that each coordinate is an integer between 1 and $n$. Notice that $S_p$ acts on $X$ via

$$\sigma \ast (a_1, a_2, \ldots, a_{p-1}, a_p) = (a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(p-1)}, a_{\sigma(p)}).$$

Let $H = \langle (1 \ 2 \ \ldots \ p) \rangle \subseteq S_p$, so $|H| = p$. Since $H$ is a subgroup of $S_p$, we know that $H$ acts on $X$ via the above action as well. For example,

$$(1 \ 2 \ \ldots \ p) \ast (a_1, a_2, \ldots, a_{p-1}, a_p) = (a_2, a_3, \ldots, a_p, a_1),$$

so $(1 \ 2 \ \ldots \ p)$ cyclically shifts an element in $X$ to the left by 1. Similarly, $(1 \ 2 \ \ldots \ p)^2$ cyclically shifts to the left by 2, etc.

a. Show that every orbit of the action of $H$ on $X$ has size either 1 or $p$.

b. Show that there are exactly $n$ orbits of size 1.

c. Show that $p \mid \left( np - n \right)$.

Note: This gives another proof of Fermat’s Little Theorem.

Problem 5: This problem provides another proof of Cauchy’s Theorem (so don’t use Cauchy’s Theorem in this problem!). Let $G$ be a group and suppose that $p$ is a prime which divides $|G|$. Let

$$X = \{(a_1, a_2, \ldots, a_{p-1}, a_p) \in G^p : a_1a_2\cdots a_{p-1}a_p = e\},$$

i.e. $X$ consists of all $p$-tuples of elements of $G$ such that when you multiply them in the given order, you obtain the identity. For example, if $G = S_3$ and $p = 3$, then $((1 \ 2), (id, (1 \ 2))) \in X$ and $((1 \ 2), (13), (1 \ 2 \ 3)) \in X$, but $(id, (1 \ 2), (123)) \notin X$.

a. Show that $|X| = |G|^{p-1}$.

b. Show that if $(a_1, a_2, \ldots, a_{p-1}, a_p) \in X$, then $(a_2, a_3, \ldots, a_p, a_1) \in X$.

Let $H = \langle (1 \ 2 \ \ldots \ p) \rangle \subseteq S_p$, so $|H| = p$. Part b says that any cyclic shift of an element of $X$ is also in $X$, so $H$ acts on $X$ as in Problem 4.

c. Notice that $|O(e, e, \ldots, e)| = 1$. Show that there exists at least one other orbit of size 1.

d. Conclude that $G$ has an element of order $p$. 