Homework 9 : Due Friday, September 17

Problem 1: Let \( n \in \mathbb{N}^+ \) and let

\[ O(n, \mathbb{R}) = \{ M \in GL_n(\mathbb{R}) : M^{-1} = M^T \} \]

Show that \( O(n, \mathbb{R}) \) is a subgroup of \( GL_n(\mathbb{R}) \). This subgroup is called the orthogonal group of degree \( n \). (Recall from linear algebra that an invertible \( n \times n \) matrix \( M \) satisfies \( M^{-1} = M^T \) if and only if the columns of \( M \) form an orthonormal basis of \( \mathbb{R}^n \). These matrices are called orthogonal matrices and they give the distance-preserving linear transformations of \( \mathbb{R}^n \).)

Problem 2: Let \( G \) be a group, and let \( H \) and \( K \) be subgroups of \( G \).

a. Show that \( H \cap K \) is a subgroup of \( G \).

b. Show, by giving an explicit counterexample, that it need not be the case that \( H \cup K \) is a subgroup of \( G \).

Cultural Note: Since \( SL_n(\mathbb{R}) \) and \( O(n, \mathbb{R}) \) are both subgroups of \( GL_n(\mathbb{R}) \), it follows from the first part that their intersection is also a subgroup of \( GL_n(\mathbb{R}) \). This subgroup is denoted by \( SO(n, \mathbb{R}) \) and is called the special orthogonal group of degree \( n \). Recall that an orthogonal matrix always has determinant \( \pm 1 \), so \( SO(n, \mathbb{R}) \) simply throws out those elements of \( O(n, \mathbb{R}) \) which have determinant \( -1 \). The elements of \( SO(n, \mathbb{R}) \) give the linear transformations of \( \mathbb{R}^n \) which are both distance-preserving and orientation-preserving.

Problem 3: Given a group \( G \), we let

\[ Z(G) = \{ a \in G : a g = g a \ \forall \ g \in G \} \]

That is, \( Z(G) \) is the set of element of \( G \) which commute with every element of \( G \). Show that \( Z(G) \) is an abelian subgroup of \( G \).

Problem 4: Let \( G \) be group and let \( a, g \in G \). The element \( gag^{-1} \) is called a conjugate of \( a \).

a. Show that \((gag^{-1})^n = ga^ng^{-1}\) for all \( n \in \mathbb{Z} \). You should start by giving a careful inductive argument for \( n \in \mathbb{N} \).

b. Show that \( |gag^{-1}| = |a| \). Thus, every conjugate of \( a \) has the same order as \( a \).

c. Show that \( |ab| = |ba| \) for all \( a, b \in G \).

Problem 5: Let \( G \) be a group with even order. Show that \( G \) has an element of order 2.

Hint: You need to show that there is a nonidentity elements which is its own inverse. Think about taking the inverse of each element and see what happens.