Problem 1: Let $R$ be a commutative ring and let $a, b \in R$.
   a. Show that $a \mid b$ if and only if $\langle b \rangle \subseteq \langle a \rangle$.
   b. Let $R$ be an integral domain. Show that $\langle a \rangle = \langle b \rangle$ if and only if $a$ and $b$ are associates.

Problem 2: Let $R$ be an integral domain and let $p \in R$.
   a. Show that if $p$ is irreducible, then every associate of $p$ is irreducible.
   b. Show that if $p$ is prime, then every associate of $p$ is prime.

Problem 3: Let $R$ be a commutative ring and let $I$ and $J$ be ideals of $R$. The product of $I$ and $J$, denoted $IJ$, is the set

   \[ IJ = \{ c_1d_1 + c_2d_2 + \cdots + c_kd_k : k \in \mathbb{N}^+, c_i \in I, d_i \in J \} \]

That is, elements of $IJ$ are the finite sums of elements which are formed as the product of an element of $I$ with an element of $J$.
   a. Prove that $IJ$ is an ideal of $R$.
   b. Show that $IJ \subseteq I \cap J$.
   c. Show that if $I = \langle a \rangle$ and $J = \langle b \rangle$, then $IJ = \langle ab \rangle$.
   d. Find an example of ideals $I$ and $J$ of some commutative ring $R$ for which $IJ \not\subseteq I \cap J$.

Problem 4: Working in the ring $\mathbb{Z}[x]$, let $I$ be the ideal

   \[ I = \langle 2, x \rangle = \{ p(x) \cdot 2 + q(x) \cdot x : p(x), q(x) \in \mathbb{Z}[x] \} \]

Show that $I$ is not a principal ideal in $\mathbb{Z}[x]$. 