Problem 1: Find the order of the following elements in the given direct product.

a. \((5, 7, 44) \in \mathbb{Z}/60\mathbb{Z} \times \mathbb{Z}/18\mathbb{Z} \times \mathbb{Z}/84\mathbb{Z}\)

b. \(((1 6 4)(3 7), r^{14}) \in S_9 \times D_{20}\)

c. \(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 3) \in O(2, \mathbb{R}) \times U(\mathbb{Z}/13\mathbb{Z})\)

Problem 2: Suppose that \(p \in \mathbb{N}^+\) is an odd prime and that there exists \(a \in \mathbb{Z}\) with \(a^2 \equiv_p -1\). Show that \(p \equiv_4 1\).

Note: The converse statement is also true (if \(p\) is prime and \(p \equiv_4 1\), then there exists \(a \in \mathbb{Z}\) with \(a^2 \equiv_p -1\)), but this requires some more advanced number theory.

Problem 3: Suppose that \(H\) is a subgroup of \(D_n\) and that \(|H|\) is odd. Show that \(H\) is cyclic.

Problem 4: Suppose that \(H\) is a subgroup of a group \(G\) with \([G : H] = 2\). Suppose that \(a, b \in G\) with both \(a \notin H\) and \(b \notin H\). Show that \(ab \in H\).

Hint: Think about the four cosets \(eH, aH, bH,\) and \(abH\).

Problem 5: Suppose that \(H\) and \(G\) are groups.

a. Show that if \(G \times H\) is cyclic, then both \(G\) and \(H\) are cyclic.

b. Suppose that \(G\) and \(H\) are both finite and cyclic. Show that \(G \times H\) is cyclic if and only if \(|G|\) and \(|H|\) are relatively prime.