Problem Set 9

Problem 1: Show that

\[ 3a^4 - 4a^3b + b^4 \geq 0 \]

for all \( a, b \in \mathbb{R} \).

Problem 2: Show that it is never possible to partition a set of six consecutive integers into two subsets in such a way that the least common multiple of the number in one subset is equal to the least common multiple of the numbers in the other.

*Problem 3: Determine if there exists an infinite sequence \((a_n)\) of positive integers having all of the following properties:

- \( a_m \nmid a_n \) whenever \( m \neq n \).
- \( \gcd(a_m, a_n) > 1 \) for all \( m, n \).
- \( \gcd\{a_n : n \in \mathbb{N}\} = 1 \).

*Problem 4: Let \( n \geq 2 \) and let \( T_n \) be the number of nonempty subsets \( S \) of \( \{1, 2, 3, \ldots, n\} \) with the property that the average of the elements of \( S \) is an integer. Prove that \( T_n - n \) is always even.

*Problem 5: Suppose that the sequence \( a_1, a_2, a_3, \ldots \) satisfies \( 0 < a_n \leq a_{2n} + a_{2n+1} \) for all \( n \geq 1 \). Prove that the series \( \sum_{n=1}^{\infty} a_n \) diverges.

*Problem 6: Is there a polynomial \( P(x) \) with integer coefficients such that \( P(10) = 400 \), \( P(14) = 440 \), and \( P(18) = 520 \)?