Problem Set 11

**Problem 1:** Show that \( \log_{10} 2 \) is irrational.

**Problem 2:** Define a sequence by letting \( a_1 = a_2 = a_3 = 1 \) and letting

\[
a_n = \frac{1 + a_{n-1}a_{n-2}}{a_{n-3}}
\]

whenever \( n \geq 4 \). Show that \( a_n \in \mathbb{Z} \) for all \( n \in \mathbb{N}^+ \).

*Problem 3:* Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function with the property that \( f(x + y) = f(x) + f(y) \) for all \( x, y \in \mathbb{R} \). Show that there exists a constant \( c \) such that \( f(x) = cx \).

*Problem 4:* The squares of an 8 \times 8 chessboard are filled with the numbers \{1, 2, 3, \ldots, 64\} in such a way that each number occurs in exactly one square. Prove that there are two adjacent squares (sharing a side) where the values differ by at least 5.

*Problem 5:* Given a partition \( \pi \) of the set \( [9] = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \), let \( \pi(x) \) be the number of elements in the part containing \( x \). Prove that given any two partitions \( \pi \) and \( \pi' \) of \([9]\), there exist distinct \( x, y \in [9] \) with \( \pi(x) = \pi(y) \) and \( \pi'(x) = \pi'(y) \).

*Problem 6:* For each integer \( m \), consider the polynomial

\[
P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2
\]

For what values of \( m \) is \( P_m(x) \) the product of two nonconstant polynomials with integer coefficients?