Homework 6 : Due Monday, February 24

Problem 1: Recall that a flush in poker is a hand in which all five of your cards have the same suit. In class, we showed that there are 5,148 many flushes (including straight flushes). Suppose that you are playing a game of poker in which each 2 is a “wild card”. That is, you can take each 2 to represent any other card. For example, if you have three different hearts, the 2 of spades, and the 2 of diamonds, then this would be considered a flush because we can pretend that the two 2’s are other hearts. In this situation, how many 5-card hands can be considered to be a flush? For this count, include any hand that could be viewed as a flush even if it could be viewed as a better hand (for example, if you have three 2’s and two clubs, count that as a flush even though it can be viewed as four-of-a-kind).

Problem 2:

a. Let \( n \in \mathbb{N}^+ \) and let \( x \in \mathbb{R} \) with \( x \geq 0 \). Use the Binomial Theorem to show that \( (1 + x)^n \geq 1 + nx \).

b. Show that \( 1 \leq n \sqrt{2} \leq 1 + \frac{1}{n} \)

Cultural Aside: Using the Squeeze Theorem, it follows that \( \lim_{n \to \infty} n \sqrt{2} = 1 \).

Problem 3:

Let \( n \in \mathbb{N}^+ \). Determine (with explanation), the value of each of the following sums:

a. \( \sum_{k=0}^{n} 2^k \cdot \binom{n}{k} = \binom{n}{0} + 2 \cdot \binom{n}{1} + 4 \cdot \binom{n}{2} + 8 \cdot \binom{n}{3} + \cdots + 2^n \cdot \binom{n}{n} \).

b. \( \sum_{k=1}^{n} (-1)^{k-1} \cdot k \cdot \binom{n}{k} = \binom{n}{1} - 2 \cdot \binom{n}{2} + 3 \cdot \binom{n}{3} - 4 \cdot \binom{n}{4} + \cdots + (-1)^{n-1} \cdot n \cdot \binom{n}{n} \).

c. \( \sum_{k=2}^{n} k \cdot (k - 1) \cdot \binom{n}{k} = 2 \cdot 1 \cdot \binom{n}{2} + 3 \cdot 2 \cdot \binom{n}{3} + \cdots + n \cdot (n - 1) \cdot \binom{n}{n} \).

Problem 4:

For all \( k, n \in \mathbb{N}^+ \) with \( k \leq n \), we know that \( k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1} \) since each side counts the number of ways of selecting a committee consisting of \( k \) people, including a distinguished president of the committee, from a group of \( n \) people.

a. Let \( k, m, n \in \mathbb{N}^+ \) with \( m \leq k \leq n \). Give a combinatorial proof (i.e. argue that both sides count the same set) of the following:

\( \binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m} \)

This generalizes the above result (which is the special case where \( m = 1 \)).

b. Let \( m, n \in \mathbb{N}^+ \) with \( m \leq n \). Find a simple formula for:

\( \sum_{k=m}^{n} \binom{n}{k} \cdot \binom{k}{m} \)

Problem 5:

Let \( a_n \) be the number of subsets of \( \{1, 2, 3, \ldots, n\} \) that do not have two adjacent numbers (so when \( n = 5 \), we allow \( \{1, 4\} \) but we do not allow \( \{1, 4, 5\} \)). Notice that:
• $a_0 = 1$ because $\emptyset$ is the only possibility.
• $a_1 = 2$ because $\emptyset$ and $\{1\}$ are both included.
• $a_2 = 3$ because $\emptyset$, $\{1\}$, and $\{2\}$ are all included, but $\{1, 2\}$ is not.

a. Give a combinatorial proof that $a_n = a_{n-1} + a_{n-2}$ whenever $n \geq 2$.

b. Using part a, explain why $a_n = f_{n+2}$ for all $n \in \mathbb{N}$, where $f_k$ is the $k^{th}$ Fibonacci number.