Problem 1: Let $n \in \mathbb{N}^+$. 

a. Evaluate 

$$\sum_{k=0}^{n} 3^k \cdot c(n, k)$$

b. Evaluate 

$$\sum_{k=0}^{n} 3^k \cdot s(n, k)$$

Note: Simplify your answers as much as possible.

Problem 2: Show that if $A$ and $B$ are countable sets, then $A \times B$ is countable.

Problem 3: 

a. Recall that $\{0, 1\}^*$ is the set of all finite sequences of 0’s and 1’s (of any finite length). Show that $\{0, 1\}^*$ is countable.

b. Let $S$ be the set of all infinite sequences of 0’s and 1’s (so an element of $S$ looks like $1100101110\ldots$). Show that $S$ is uncountable.

Problem 4: Fix $n \in \mathbb{N}^+$. Consider the graph $Q_n$ defined as follows. Let the vertex set $V$ be the set of all sequences of 0’s and 1’s of length $n$ (so for example, if $n = 3$, then one vertex is 010 and another is 111). Let $E$ be the set of all pairs $\{u, v\}$ such that $u$ and $v$ differ in exactly one coordinate (so for example when $n = 3$ there is edge with endpoints 001 and 101). The graph $Q_n$ is called the $n$-cube.

a. Draw the graphs $Q_1$, $Q_2$, and $Q_3$.

b. Write down the adjacency matrix for $Q_3$ (clearly indicate the ordering of the vertices that you are using).


d. Determine $d(u)$ for each $u \in Q_n$.

e. Determine the number of vertices and edges in $Q_n$.

Problem 5: Let $G$ be a finite graph. Explain why the number of 1’s in any adjacency matrix of $G$ equals the number of 1’s in any incidence matrix of $G$.

Problem 6: Let $G$ be a finite graph with $|V| \geq 2$. Show that there exists $u, w \in V$ with $u \neq w$ such that $d(u) = d(w)$. 