Homework 9 : Due Monday, March 12

Problem 1: Let \( n \in \mathbb{N}^+ \).
   a. Evaluate \( \sum_{k=0}^{n} c(n, k) \)
   b. Evaluate \( \sum_{k=0}^{n} 2^k c(n, k) \)

Problem 2: On Homework 7, you showed that
   \[ S(n, n - 2) = \binom{n}{3} + 3 \cdot \binom{n}{4} \]
   for all \( n \geq 3 \). Now show that
   \[ c(n, n - 2) = 2 \cdot \binom{n}{3} + 3 \cdot \binom{n}{4} \]
   for all \( n \geq 3 \).

Problem 3: Let \( \ell, n \in \mathbb{N}^+ \) with \( \ell \leq n \). If \( \sigma \) is a permutation of \([n]\), then we say that \( i \in [n] \) is a fixed point of \( \sigma \) if \( \sigma(i) = i \). How many permutations of \([n]\) have exactly \( \ell \) fixed points?

Problem 4: How many different ways can you place seven distinct ornaments on three identical circular wreaths? Allow the possibility that some wreaths have no ornaments on them.

Problem 5: Let \( n \in \mathbb{N}^+ \).
   a. How many ways are there to break up \( 3n \) people into \( n \) groups of size 3 (where there is no ordering amongst the groups)? Simplify your answer as much as possible.
   b. How many permutations of \([3n]\) consist of \( n \) distinct 3-cycles?
   c. Explain why your answers in parts a and b are different.

Problem 6: Suppose that \( \sigma \) is a permutation of \([n]\). Define an \( n \times n \) matrix \( M(\sigma) \) by letting
   \[ M(\sigma)_{i,j} = \begin{cases} 1 & \text{if } \sigma(j) = i \\ 0 & \text{otherwise} \end{cases} \]
   a. Let \( n = 4 \), let \( \sigma = (1 2 3)(4) \) and let \( \tau = (1 2)(3 4) \). Write down \( M(\sigma) \), \( M(\tau) \), and \( M(\sigma \circ \tau) \).
   b. Show that \( M(\sigma \circ \tau) = M(\sigma) \cdot M(\tau) \) for all permutations \( \sigma \) and \( \tau \) of \([n]\) (not just those in part a).