Problem 1: Recall that a flush in poker is a hand in which all five of your cards have the same suit. In class, we showed that there are 5,148 many flushes (including straight flushes). Suppose that you are playing a game of poker in which each 2 is a “wild card”. That is, you can take each 2 to represent any other card. For example, if you have three different hearts, the 2 of spades, and the 2 of diamonds, then this would be considered a flush because we can pretend that the two 2’s are other hearts. In this situation, how many 5-card hands can be considered to be a flush? For this count, include any hand that could be viewed as a flush even if it could be viewed as a better hand (for example, if you have three 2’s and two clubs, count that as a flush even though it can be viewed as four-of-a-kind).

Problem 2:
a. Use the Binomial Theorem to show that \((1 + x)^n \geq 1 + nx\) whenever \(n \in \mathbb{N}\) and \(x \in \mathbb{R}\) with \(x \geq 0\).
b. Use induction to show that \((1 + x)^n \geq 1 + nx\) whenever \(n \in \mathbb{N}\) and \(x \in \mathbb{R}\) with \(x \geq -1\). Clearly explain where you are using the assumption that \(x \geq -1\).

c. Determine (with explanation) the value of

\[
\sum_{k=0}^{n} 2^k \cdot \left( \begin{array}{c} n \\ k \end{array} \right) = \left( \begin{array}{c} n \\ 0 \end{array} \right) + 2 \cdot \left( \begin{array}{c} n \\ 1 \end{array} \right) + 4 \cdot \left( \begin{array}{c} n \\ 2 \end{array} \right) + 8 \cdot \left( \begin{array}{c} n \\ 3 \end{array} \right) + \cdots + 2^n \cdot \left( \begin{array}{c} n \\ n \end{array} \right)
\]

Problem 3: Let \(n \in \mathbb{N}^+\).
a. Determine (with explanation) the value of

\[
\sum_{k=0}^{n} 2^k \cdot \left( \begin{array}{c} n \\ k \end{array} \right) = \left( \begin{array}{c} n \\ 0 \end{array} \right) + 2 \cdot \left( \begin{array}{c} n \\ 1 \end{array} \right) + 4 \cdot \left( \begin{array}{c} n \\ 2 \end{array} \right) + 8 \cdot \left( \begin{array}{c} n \\ 3 \end{array} \right) + \cdots + 2^n \cdot \left( \begin{array}{c} n \\ n \end{array} \right)
\]

b. Determine (with explanation) the value of

\[
\sum_{k=1}^{n} (-1)^{k-1} \cdot k \cdot \left( \begin{array}{c} n \\ k \end{array} \right) = \left( \begin{array}{c} n \\ 1 \end{array} \right) - 2 \cdot \left( \begin{array}{c} n \\ 2 \end{array} \right) + 3 \cdot \left( \begin{array}{c} n \\ 3 \end{array} \right) - 4 \cdot \left( \begin{array}{c} n \\ 4 \end{array} \right) + \cdots + (-1)^{n-1} \cdot n \cdot \left( \begin{array}{c} n \\ n \end{array} \right)
\]

c. Determine (with explanation) the value of

\[
\sum_{k=2}^{n} k \cdot (k-1) \cdot \left( \begin{array}{c} n \\ k \end{array} \right) = 2 \cdot \left( \begin{array}{c} n \\ 2 \end{array} \right) + 3 \cdot \left( \begin{array}{c} n \\ 3 \end{array} \right) + \cdots + n \cdot (n-1) \cdot \left( \begin{array}{c} n \\ n \end{array} \right)
\]

Problem 4: For any \(k, n \in \mathbb{N}^+\) with \(k \leq n\), we know that \(k \cdot \left( \begin{array}{c} n \\ k \end{array} \right) = n \cdot \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right)\) since each side counts the number of ways of selecting a committee consisting of \(k\) people, including a distinguished president of the committee, from a group of \(n\) people.
a. Let \(k, m, n \in \mathbb{N}^+\) with \(m \leq k \leq n\). Give a combinatorial proof (i.e. argue that both sides count the same set) of the following:

\[
\left( \begin{array}{c} n \\ k \end{array} \right) \cdot \left( \begin{array}{c} k \\ m \end{array} \right) = \left( \begin{array}{c} n \\ m \end{array} \right) \cdot \left( \begin{array}{c} n-m \\ k-m \end{array} \right)
\]

This generalizes the above result (the above is the special case where \(m = 1\)).
b. Let \(m, n \in \mathbb{N}^+\) with \(m \leq n\). Find a simple formula for:

\[
\sum_{k=m}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \cdot \left( \begin{array}{c} k \\ m \end{array} \right)
\]

Problem 5: Let \(n \in \mathbb{N}^+\). We know from class that

\[
1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}
\]
Notice the right-hand side is simply \((n+1)\). Give a direct combinatorial proof that

\[
1 + 2 + 3 + \cdots + n = \binom{n + 1}{2}
\]

by arguing that each side counts the number of ordered pairs \((k, \ell)\) with \(k, \ell \in \mathbb{N}\) and \(0 \leq k < \ell \leq n\).