Written Assignment 4: Due Wednesday, October 8

**Problem 1:** Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a surjective linear transformation and that $\vec{u}_1, \vec{u}_2 \in \mathbb{R}^2$. Show that if $\text{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2$, then $\text{Span}(T(\vec{u}_1), T(\vec{u}_2)) = \mathbb{R}^2$.

*Hint:* We want to show that $\mathbb{R}^2 \subseteq \text{Span}(T(\vec{u}_1), T(\vec{u}_2))$. Start by taking an arbitrary $\vec{w} \in \mathbb{R}^2$. To show that $\vec{w} \in \text{Span}(T(\vec{u}_1), T(\vec{u}_2))$, what do you need to do?

**Problem 2:** Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an injective linear transformation and that $\vec{u}, \vec{w} \in \mathbb{R}^2$. Show that if $\vec{w} \notin \text{Span}(\vec{u})$, then $T(\vec{w}) \notin \text{Span}(T(\vec{u}))$.

**Problem 3:** In this problem, we determine which $2 \times 2$ matrices commute with *every* $2 \times 2$ matrix.

a. Show that if $r \in \mathbb{R}$ and we let

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

then $AB = BA$ for every $2 \times 2$ matrix $B$.

b. Suppose that $A$ is a $2 \times 2$ matrix with the property that $AB = BA$ for every $2 \times 2$ matrix $B$. Show that there exists $r \in \mathbb{R}$ such that

$$A = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

*Hint:* For part b, make strategic choices for $B$ to make your life as simple as possible. I suggest thinking about matrices with lots of zeros.