Problem Set 6: Due Monday, September 22

Note: In Problems 1 and 4, please underline or write in a different color the parts that go into the blanks.

Problem 1: Fill in the blanks below with appropriate phrases so that the result is a correct proof of the statement following statement: If \( \vec{u}, \vec{w} \in \mathbb{R}^2 \) and \( \vec{w} \in \text{Span}(\vec{u}) \), then \( \text{Span}(\vec{u}) \subseteq \text{Span}(\vec{w}) \).

Let \( \vec{v} \in \text{Span}(\vec{w}) \) be arbitrary. Since \( \vec{w} \in \text{Span}(\vec{u}) \), we can \( \vec{v} = \lambda \vec{w} \). Since \( \lambda \in \mathbb{R} \), we conclude that \( \vec{v} \in \text{Span}(\vec{u}) \) since \( \vec{v} = \lambda \vec{w} \). Now notice that \( \vec{v} = \lambda \vec{w} \). Since \( \lambda \in \mathbb{R} \), we conclude that \( \vec{v} \in \text{Span}(\vec{u}) \). Since \( \vec{v} \in \text{Span}(\vec{u}) \) was arbitrary, the result follows.

Problem 2: Given \( \vec{u} \in \mathbb{R}^2 \), is the set \( \text{Span}(\vec{u}) \) always closed under componentwise multiplication? In other words, if

\[
\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in \text{Span}(\vec{u}) \quad \text{and} \quad \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in \text{Span}(\vec{u}),
\]

must it be the case that

\[
\begin{pmatrix} a_1 a_2 \\ b_1 b_2 \end{pmatrix} \in \text{Span}(\vec{u})?
\]

Either argue that this is always true, or provide a specific counterexample (with justification).

Problem 3: Let \( \vec{u}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \) and let \( \vec{u}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \).

a. Show that \( \text{Span}(\vec{u}_1, \vec{u}_2) = \mathbb{R}^2 \).

b. Find the coordinates of \( \begin{pmatrix} 5 \\ 1 \end{pmatrix} \) relative to \( \vec{u}_1 \) and \( \vec{u}_2 \). In other words, calculate \( \text{Coord}(\vec{u}_1, \vec{u}_2) \left( \begin{pmatrix} 5 \\ 1 \end{pmatrix} \right) \).

c. Find the coordinates of \( \begin{pmatrix} 8 \\ 17 \end{pmatrix} \) relative to \( \vec{u}_1 \) and \( \vec{u}_2 \). In other words, calculate \( \text{Coord}(\vec{u}_1, \vec{u}_2) \left( \begin{pmatrix} 8 \\ 17 \end{pmatrix} \right) \).

In each part, briefly explain how you carried out your computation.

Problem 4: In this problem we work through the proof of Proposition 2.9 in the notes, which says the following: Let \( \vec{u}_1, \vec{u}_2 \in \mathbb{R}^2 \). The following are equivalent.

1. \( \text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1) \).

2. \( \vec{u}_2 \in \text{Span}(\vec{u}_1) \).

Fill in the blanks below with appropriate phrases so that the result is a correct proof:

We first show that 1 implies 2. Assume then that 1 is true, so assume that \( \text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1) \). Notice that \( \vec{u}_2 = \lambda \vec{u}_1 \). Since \( \lambda \in \mathbb{R} \), it follows that \( \vec{u}_2 \in \text{Span}(\vec{u}_1) \). Since \( \text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1) \), we conclude that \( \vec{u}_2 \in \text{Span}(\vec{u}_1) \).

We now show that 2 implies 1. Assume then that 2 is true, so assume that \( \vec{u}_2 \in \text{Span}(\vec{u}_1) \). By definition, we can \( \vec{u}_2 = \lambda \vec{u}_1 \). To show that \( \text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1) \), we give a double containment proof.

- Using Proposition \( \text{Proposition 2.9} \), we know immediately that \( \text{Span}(\vec{u}_1) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2) \).

- We now show that \( \text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1) \). Let \( \vec{v} \in \text{Span}(\vec{u}_1, \vec{u}_2) \) be arbitrary. By definition we can \( \vec{v} = \lambda \vec{u}_1 + \mu \vec{u}_2 \). Notice that \( \vec{v} = \lambda \vec{u}_1 + \mu \vec{u}_2 \). Since \( \lambda, \mu \in \mathbb{R} \), it follows that \( \vec{v} \in \text{Span}(\vec{u}_1) \). Since \( \vec{v} \in \text{Span}(\vec{u}_1) \) was arbitrary, we conclude that \( \text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1) \).

Since we have shown both \( \text{Span}(\vec{u}_1) \subseteq \text{Span}(\vec{u}_1, \vec{u}_2) \) and \( \text{Span}(\vec{u}_1, \vec{u}_2) \subseteq \text{Span}(\vec{u}_1) \), we conclude that \( \text{Span}(\vec{u}_1, \vec{u}_2) = \text{Span}(\vec{u}_1) \).