Problem Set 10: Due Friday, October 10

Problem 1: Consider the unique linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ with

$$[T] = \begin{pmatrix} 2 & -5 \\ -6 & 15 \end{pmatrix}$$

Find, with explanation, vectors $\vec{u}, \vec{w} \in \mathbb{R}^2$ with $\text{Null}(T) = \text{Span}(\vec{u})$ and $\text{range}(T) = \text{Span}(\vec{w})$.

Problem 2: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Recall that

$\text{Null}(T) = \{ \vec{v} \in \mathbb{R}^2 : T(\vec{v}) = \vec{0} \}$.

a. Show that if $\vec{v}_1, \vec{v}_2 \in \text{Null}(T)$, then $\vec{v}_1 + \vec{v}_2 \in \text{Null}(T)$.

b. Show that if $\vec{v} \in \text{Null}(T)$ and $c \in \mathbb{R}$, then $c \cdot \vec{v} \in \text{Null}(T)$.

Problem 3: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the unique linear transformation with

$$[T] = \begin{pmatrix} 7 & -9 \\ -3 & 4 \end{pmatrix}$$

Explain why $T$ is invertible and calculate $T^{-1}\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Problem 4: Consider the following system of equations:

$$\begin{align*}
x + 4y &= -3 \\
2x + 5y &= 8
\end{align*}$$

a. Rewrite the above system in the form $A\vec{v} = \vec{b}$ for some matrix $A$ and vector $\vec{b}$.

b. Explain why $A$ is invertible and calculate $A^{-1}$.

c. Use $A^{-1}$ to solve the system.

Problem 5: In this problem, let $0$ denote the $2 \times 2$ zero matrix, i.e. the $2 \times 2$ matrix where all four entries are 0.

a. Give an example of a nonzero $2 \times 2$ matrix $A$ with $A \cdot A = 0$.

b. Show that if $A$ is invertible and $A \cdot A = 0$, then $A = 0$.

Note: Since $0$ is not invertible, it follows from part b that there is no invertible matrix $A$ with $A \cdot A = 0$.

Problem 6: Let $A, B, C$ all be invertible $2 \times 2$ matrices. Must there exist a $2 \times 2$ matrix $X$ with

$$A(X + B)C = I?$$

Either justify carefully or give a counterexample.