IMMERSE 2005
Analysis Problem Set 7

1. Prove that for all x, y, z in an inner product space $(V, \langle \cdot, \cdot \rangle)$ and all $\alpha \in \mathbb{C}$, we have

$$\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle \text{ and } \langle x, \alpha y \rangle = \overline{\alpha} \langle x, y \rangle$$

In other words, the inner product is conjugate linear in the second entry.
Solution:
$\langle x, y+z \rangle = \overline{\langle y+z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle} = \langle x, y \rangle + \langle x, z \rangle$

$\langle x, \alpha y \rangle = \overline{\langle \alpha y, x \rangle} = \overline{\alpha \langle y, x \rangle} = \overline{\alpha}\,\overline{\langle y, x \rangle} = \overline{\alpha} \langle x, y \rangle$

2. If V has an inner product $\langle \cdot, \cdot \rangle$, prove that $\|x\| = \langle x, x \rangle^{1/2}$ defines a norm on V.

Proof. $\langle x, x \rangle > 0$ for all $x \in V \setminus \{0\}$ and $\langle 0, 0 \rangle = 0$ follows from the definition.
Let $\alpha \in \mathbb{C}$ and $x, y \in V$. Then

$$\|\alpha x\| = \langle \alpha x, \alpha x \rangle^{1/2} = (\alpha \overline{\alpha} \langle x, x \rangle)^{1/2} = |\alpha| \|x\|$$

and

$$\|x+y\| = \langle x+y, x+y \rangle^{1/2} = \left(\|x\|^2 + 2\mathrm{Re}\langle x, y \rangle + \|y\|^2 \right)^{1/2} \leq \left(\|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \right)^{1/2} = \|x\| + \|y\|$$

□

3. We want to prove that a norm on a vector space V is generated from an inner product if and only if the norm satisfies the Parallelogram Identity:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall\ x, y \in V.$$

(a) Prove the forward direction: Let V have an inner product and show that the norm $\|x\|^2 = \langle x, x \rangle$ satisfies the Parallelogram Identity.

Let V have an inner product, and suppose $\|x\|^2 = \langle x, x \rangle$. Then, we have

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x+y, x+y \rangle + \langle x-y, x-y \rangle \\
&= (\langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle) + (\langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle) \\
&= 2\langle x, x \rangle + 2\langle y, y \rangle \\
&= 2\|x\|^2 + 2\|y\|^2,
\end{aligned}$$

so in particular, $\| \cdot \|$ satisfies the Parallelogram Identity.

(b) Prove the reverse direction for V a real vector space. Don't forget to adjust the properties of an inner product on a real vector space.

Suppose the real vector space V is endowed with an inner product $\| \cdot \|$ that satisfies the Parallelogram Identity. Let

$$\langle x, y\rangle := \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2}.$$

Then, we have $\langle x, x\rangle = 0$, and $\langle x, y\rangle = \langle y, x\rangle$ for each $x, y \in V$.

We now that $\langle x, y\rangle$ is linear in the first coordinate.

Using the definition of $\langle x, y\rangle$, we have

$$\begin{aligned} 2\left(\langle x+y, z\rangle - \langle x, z\rangle - \langle y, z\rangle\right) &= \underbrace{\|(x+y)+z\|^2}_{(\star)} - \underbrace{\|x+y\|^2}_{(\star\star)} - \|z\|^2 - \|x+z\|^2 + \|x\|^2 + \|z\|^2 \\ &\quad - \underbrace{\|y+z\|^2}_{(\star\star\star)} + \|y\|^2 + \|z\|^2. \end{aligned} \tag{1}$$

Now, we expand the starred components of (**??**), using the Parallelogram Identity:

$$(\star) \qquad \begin{aligned} \|(x+y)+z\|^2 &= \|(x+y/2)+(y/2+z)\|^2 \\ &= -\|x-z\|^2 + 2\|x+y/2\|^2 + 2\|y/2+z\|^2, \end{aligned}$$

$$(\star\star) \qquad \begin{aligned} \|x+y\|^2 &= \|(x+y/2)+y/2\|^2 \\ &= -\|x\|^2 + 2\|x+y/2\|^2 + 2\|y/2\|^2, \end{aligned}$$

and

$$(\star\star\star) \qquad \begin{aligned} \|y+z\|^2 &= \|(y/2)+(y/2+z)\|^2 \\ &= \|z\|^2 + 2\|y/2\|^2 + 2\|y/2+z\|^2. \end{aligned}$$

Substituting these back into (**??**), and cancelling terms, we arrive at

$$\begin{aligned} 2\left(\langle x+y, z\rangle - \langle x, z\rangle - \langle y, z\rangle\right) &= -\|x-z\|^2 + 2\|x\|^2 - \|x+z\|^2 + 2\|z\|^2 \\ &= 0, \end{aligned}$$

again by the Parallelogram Identity. Hence, we conclude that $\langle x+y, z\rangle - \langle x, z\rangle - \langle y, z\rangle = 0$ for each set of vectors $x, y, z \in V$, or

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

so that $\langle \cdot, \cdot\rangle$ is linear in the first coordinate.

Finally, we must show that $\langle \alpha x, y\rangle = |\alpha|\langle x, y\rangle$ when $\alpha \in \mathbb{R}$.

Let $x, y \in V$, and notice that by linearity, we can say that for $n \in \mathbb{N}$, $\langle nx, y\rangle = n\langle x, y\rangle$. Now, we have:

$$\begin{aligned} \langle -x, y\rangle &= \langle (1-2)x, y\rangle \\ &= \langle x-2x, y\rangle \\ &= \langle x, y\rangle + \langle -2x, y\rangle \\ &= \langle x, y\rangle + 2\langle -x, y\rangle, \end{aligned}$$

so in particular, we have $\langle -x, y\rangle = \langle x, y\rangle + 2\langle -x, y\rangle$, so that $-\langle x, y\rangle = \langle -x, y\rangle$. Thus, for $z \in \mathbb{Z}$, we have $\langle zx, y\rangle = z\langle x, y\rangle$.

Now, let $q = \dfrac{a}{b} \in \mathbb{Q}$. Then,

$$\begin{aligned}\langle qx, y\rangle &= \langle ax/b, y\rangle \\ &= \frac{\|ax/b + y\|^2 - \|ax/b\|^2 - \|y\|^2}{2} \\ &= \frac{\frac{1}{b^2}\|ax + by\|^2 - \frac{1}{b^2}\|ax\|^2 - \frac{1}{b^2}\|by\|^2}{2} \\ &= \frac{1}{b^2}\langle ax, by\rangle.\end{aligned}$$

Thus, we have

$$\begin{aligned}\langle ax/b, y\rangle &= \frac{1}{b^2}\langle ax, by\rangle \\ &= \frac{a}{b^2}\langle x, by\rangle \\ &= \frac{a}{b^2}\langle by, x\rangle \\ &= \frac{ab}{b^2}\langle y, x\rangle \\ &= \frac{a}{b}\langle x, y\rangle.\end{aligned}$$

Now, let $r \in \mathbb{R}$ be given, and $\{q_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$ such that $\lim_{n\to\infty} q_n = r$. Then, we have

$$\begin{aligned}|r\langle x, y\rangle - \langle rx, y\rangle| &\leq |r\langle x, y\rangle - q_n\langle x, y\rangle| + |\langle (r - q_n)x, y\rangle| \\ &\leq |r - q_n|(|\langle x, y\rangle| + \|x\|\|y\|),\end{aligned}$$

using the argument above and Cauchy-Schwartz. The last term in the inequality above can be made arbitrarily small, yielding the result.

4. Prove that the sup norm on $C[a, b]$ does not come from an inner product.

Proof. If the sup norm came from an inner product, then it would satisfy the Parallelogram Identity. However with $f(x) = 1$ and $g(x) = \dfrac{x - a}{b - a}$, we have

$$\|f + g\|^2 + \|f - g\|^2 - 2\|f\|^2 - 2\|g\|^2 = 4 + 1 - 2 - 2 \neq 0.$$

□

5. Determine whether the Parallelogram Identity is satisfied in $(\ell^1, \|\cdot\|_1)$ or in $(\ell^\infty, \|\cdot\|_\infty)$.

The Parallelogram Identity is satisfied in neither of these spaces. To show this consider $x = (1, 0, 0, 0, ...)$ and $y = (0, 1, 0, 0, ...)$ in $\ell^1 \subset \ell^\infty$.

$$\|x + y\|_\infty^2 + \|x - y\|_\infty^2 - 2\|x\|_\infty^2 - 2\|y\|_\infty^2 = 1 + 1 - 2 - 2 \neq 0$$

and

$$\|x+y\|_1^2 + \|x-y\|_1^2 - 2\|x\|_1^2 - 2\|y\|_1^2 = 4+4-2-2 \neq 0.$$

6. Let V be an inner product space, and S a subset of V. Prove that $S^\perp$ is a subspace of V.

We verify that $S^\perp$ is closed under vector addition and scalar multiplication.

Let $w, v \in S^\perp$, so $\langle w, s\rangle = 0 = \langle v, s\rangle$ for each $s \in S$. Then, $\langle w+v, s\rangle = \langle w, s\rangle + \langle v, s\rangle = 0+0 = 0$ for all $s \in S$; hence, $w+v \in S^\perp$.

Finally, if $w \in S^\perp$, then $\langle \alpha w, s\rangle = \alpha\langle w, s\rangle = \alpha \cdot 0 = 0$ for all $\alpha \in \mathbb{C}$, so $\alpha w \in S^\perp$ for all $\alpha \in \mathbb{C}$.

7. Prove that the inner product on a Hilbert space is continuous in each entry. In other words, show that for each $y \in V$, $f_y(x) = \langle y, x\rangle$ and $g_y(x) = \langle x, y\rangle$ are continuous functions from V to $\mathbb{C}$.

Solution: We will do the proof for g_y. The proof for f_y is very similar.

Let $v \in V$, and $\epsilon > 0$. Let $\delta = \dfrac{\epsilon}{\|y\|}$. If $\|v-a\| < \delta$ then $|g_y(v) - g_y(a)| = |\langle v, y\rangle - \langle a, y\rangle| = |\langle v-a, y\rangle|$. By Cauchy-Schwartz, $|\langle v-a, y\rangle| \leq \|v-a\|\,\|y\| < \dfrac{\epsilon}{\|y\|}\|y\| = \epsilon$.

8. Prove that the trigonometric system

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \frac{\sin(kx)}{\sqrt{\pi}}\right\}_{k=1}^{\infty}$$

is an orthonormal system in $L^2[-\pi,\pi]$.

First we have $\left\langle \dfrac{1}{\sqrt{2\pi}}, \dfrac{1}{\sqrt{2\pi}} \right\rangle = 1$. To simplify this problem note that $\sin(kx)$ and $\cos(kx)$ go through k periods on the interval $[-\pi,\pi]$, so they integrate to zero on this interval. Therefore

$$\left\langle \frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}} \right\rangle = \frac{1}{\pi\sqrt{2}} \int_{-\pi}^{\pi} \cos(kx)dx = 0$$

and similarly we have $\left\langle \dfrac{1}{\sqrt{2\pi}}, \dfrac{\sin(kx)}{\sqrt{\pi}} \right\rangle = 0$.

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(kx)\cos(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((k-n)x) + \cos((k+n)x)dx \\
&= \pi\delta_{k,n} \\
\int_{-\pi}^{\pi} \sin(kx)\sin(nx)dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((k-n)x) - \cos((k+n)x)dx \\
&= \pi\delta_{k,n}
\end{aligned}$$

Finally $\left\langle \dfrac{\cos(kx)}{\sqrt{\pi}}, \dfrac{\sin(nx)}{\sqrt{\pi}} \right\rangle = 0$ since $\sin(kx)\cos(nx)$ is an odd function, and we are integrating on a symmetric domain.

9. Let V be a Hilbert space. Prove that $\langle x, y\rangle = 0$ for all $y \in V$ if and only if $x = 0$.

($\Rightarrow$) If $\langle x, y\rangle = 0$ for all $y \in V$, then $\langle x, x\rangle = 0$, implying $x = 0$.

($\Leftarrow$) If $x = 0$, then $\langle 0, y \langle = 0 \langle x, y \rangle = 0$, for all $y \in V$ (with x any non-zero vector in V).

10. Assume $\{f_n\}_{n=1}^{\infty}$ converges to f in $L^2[a,b]$. Prove that $\{\|f_n\|_2\}_{n=1}^{\infty}$ is a bounded sequence of real numbers.

Proof. Since $f_n \to f$ in $L^2[a,b]$, we have that for $\epsilon = 1$, there is an $N \in \mathbb{N}$ for which $n \geq N$ implies $\|f_n - f\|_2 < 1$. Using the reverse triangle inequality, we have that $\|f_n\|_2 - \|f\|_2 < 1$, or $\|f_n\|_2 < 1 + \|f\|_2$ for $n \geq N$.

Letting $M := \max\{\|f_1\|_2, \ldots, \|f_{N-1}\|_2, \|f_N\|_2 + 1\}$, we see that $\|f_i\|_2 \leq M$ for all $i \in \mathbb{N}$. Thus, $\{\|f_n\|_2\}_{n=1}^{\infty}$ is a bounded sequence of reals. □