IMMERSE 2005
Analysis Problem Set 5

1. Prove that the three characterizations given in class of a continuous function are equivalent.

Recall that the characterizations are:

(1) $f : (X, \rho) \to (Y, d)$ is continuous on X (this is the ϵ-δ definition).

(2) $f^{-1}(V) = \{x \in X \ : \ f(x) \in V\}$ is open in X for each open set V in Y.

(3) For each sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ converging to $x \in X$, we have $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in Y$.

Proof. [(1) $\Rightarrow$ (2)] Suppose that f is continuous, and let U be open in Y. Fix $a \in f^{-1}(U)$. Since U is open in Y, and since U contains $u = f(a)$, there is an $\epsilon > 0$ for which $B_\epsilon(f(a)) \subseteq U$. Since f is continuous at a, there is a $\delta > 0$ for which $\rho(x, a) < \delta$ implies $d(f(x), f(a)) < \epsilon$. *I.e.*, $f(B_\delta(a) \cap X) \subseteq B_\epsilon(f(a)) \subseteq U$. So, $f^{-1}(U)$ contains $B_\delta(a) \cap X$. In particular, $f^{-1}(U)$ contains an open ball about a; since a was arbitrary in $f^{-1}(U)$, we have that $f^{-1}(U)$ is open.

[(2) $\Rightarrow$ (1)] Fix $a \in X$, and $\epsilon > 0$. Using $U = B_\epsilon(f(a))$, we have that $f^{-1}(U)$ is open in X and contains a. So there is a $\delta > 0$ such that $B_\delta(a) \cap X \subseteq f^{-1}(U)$. *I.e.*, $\rho(x, a) < \delta$ implies $d(f(x), f(a)) < \epsilon$. Thus, f is continuous at a.

[(1) $\Rightarrow$ (3)] Suppose f is continuous on X, fix $\epsilon > 0$, and suppose that $\{x_n\}_{n=1}^{\infty}$ converges to x in X. Since f is continuous at x, we have that $\rho(a, x) < \delta$ implies $d(f(a), f(x)) < \epsilon$. Now, there is an $N \in \mathbb{N}$ for which $n > N$ implies $\rho(x_n, x) < \delta$, in turn implying $d(f(x_n), f(x)) < \epsilon$. Thus, $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x)$ in (Y, d).

[(3) $\Rightarrow$ (1)] Suppose that (1) is false, and fix $a \in X$. So there is an $\epsilon > 0$ such that for each $\delta > 0$, there is an $x \in X$ with $\rho(x, a) < \delta$ yet $d(f(x), f(a)) \geq \epsilon$. Taking $\delta = \dfrac{1}{n}$, there is an $x_n \in X$ with $\rho(x_n, a) < \dfrac{1}{n}$, yet $d(f(x_n), f(a)) \geq \epsilon$. So $\{x_n\}_{n=1}^{\infty}$ converges to a in (X, ρ), yet $\{f(x_n)\}_{n=1}^{\infty}$ fails to converge to $f(a)$ in (Y, d). Thus, (3) is false.

$\square$

2. Let (X, ρ) be a metric space and let $(V, \|\cdot\|)$ be a normed vector space. Let $C_b(X, V)$ be the vector space of all bounded continuous functions $f : X \to V$. Prove that

$$\|f\|_\infty = \sup\{\|f(x)\|_V : x \in X\}$$

is a norm on $C_b(X, V)$.

The only property worth showing is the triangle inequality. Let $f, g \in C_b(X, V)$.

$$\begin{aligned}\|f + g\|_\infty &= \sup\{\|f(x) + g(x)\|_V : x \in X\} \leq \sup\{\|f(x)\|_V + \|g(x)\|_V : x \in X\} \\ &\leq \sup\{\|f(x)\|_V : x \in X\} + \sup\{\|g(x)\|_V : x \in X\} = \|f\|_\infty + \|g\|_\infty.\end{aligned}$$

The other properties follow directly from properties of the norm on V.

3. For each of the following statements, prove the statement or find a counterexample which disproves it. In each part, f is a continuous function from a metric space (X, ρ) to a metric space (Y, σ).

(a) If C is compact in X, then $f(C)$ is compact in Y.

(b) If C is connected, then $f(C)$ is connected in Y.

(c) If C is path connected in X, then $f(C)$ is path connected in Y.

Solution:

(a) This is true: Let $\{y_n\}$ be a sequence in $f(C)$. We will show that this sequence contains a convergent subsequence:

Let $\{x_n\}$ be such that for all $i \in \mathbb{N}$ $x_i \in C$, and $f(x_i) = y_i$. Then since C is compact, there is a subsequence $\{x'_n\} \subseteq C$ converging to $x \in C$. Consider $\{y'_n\} = \{f(x'_n)\}$, a subsequence of $\{y_n\}$. Since f is continuous, $\lim_{n\to\infty} y'_n = \lim_{n\to\infty} f(x'_n) = f(x) \in f(C)$. Thus $\{y_n\}$ has a convergent subsequence, and we're done.

(b) This is true: Say $f(C)$ is disconnected, then there is a separation A, B of $f(C)$.

Consider $A' = f^{-1}(A) \cap C$, $B' = f^{-1}(B) \cap C$. We will show that these form a separation of C, establishing a contradiction.

$C \subseteq f^{-1}(f(C)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. Thus $A' \cup B' = C$.

$\overline{A'} \cap B' = \emptyset$: If $a' = \lim_{n\to\infty} a'_n$ is a limit point of A' then since f is continuous we have $a := \lim_{n\to\infty} f(a'_n) = f(a')$ is a limit point of A. Since $\overline{A} \cap B = \emptyset$, $a \notin B$. Thus $a' \notin f^{-1}(B)$. Thus $\overline{A'} \cap B' = \emptyset$. Similarly, $A' \cap \overline{B'} = \emptyset$

(c) This is true. Let $f(a), f(b) \in f(C)$ where $a, b \in C$. Since C is path connected, there is a continuous path $p : [0,1] \to C$ with $p(0) = a$, and $p(1) = b$. Composition of continuous function is continuous, so $g := f \circ p$ is a continuous function from $[0,1]$ to $f(C)$. Furthermore, $g(0) = f(p(0)) = f(a)$, and $g(1) = f(p(1)) = f(b)$. Thus any two points in $f(C)$ are connected by a continuous path through $f(C)$, and we're done.

4. Let $A \subset X$ be path connected. Is $\overline{A}$ necessarily path connected?

Solution: No. Let A be the following subset of $\mathbb{R}^2$: $A = \{(x, \sin\frac{1}{x}) \ : \ x \in (0, 2\pi]\}$. Then, since A is generated by the image of $f(x) = \sin\frac{1}{x}$ on $(0, \pi]$, and since $f(x)$ is continuous on such an interval, A is path-connected.

However, we have

$$\overline{A} = A \bigcup \{(0, y) \ : \ y \in [-1, 1]\}.$$

So, $\overline{A}$ is the topologist's sine curve, and is not path-connected. To see that this is so, suppose the contrary were true. Then there would exist a path between $(0,0)$ and $(1, \sin(1)) = u$. So there is a continuous function $\gamma(t) = (x(t), y(t)) \ : \ [0,1] \to \mathbb{R}^2$ such that $f(0) = u$ and $f(1) = (0,0)$. Choose $t_1 \in (0,1)$ such that $y(t_1) = -1$ and $x(t_1) < 1$. Choose $t_2 > t_1$ in $(1/2, 1)$ such that $y(t_2) = 1$ and $x(t_2) < \frac{1}{2}$. Iterating this procedure, choose $t_n > t_{n+1}$ in $(\frac{n-1}{n}, 1)$ with $y(t_n) = (-1)^n$ and $x(t_n) < \frac{1}{n}$. Then as $n \to \infty$, $x(t_n) \to 0$ yet $y(t_n)$ fails to converge. Since $f(t) = (x(t), y(t))$ is continuous on $[0, 1]$ if and only if $x(t)$ and $y(t)$ are both continuous on $[0, 1]$, we see that because of the $\{y(t_n)\}$, $f(t)$ is not continuous at $t = 1$, since $f(1) = (0, 0)$.

Thus, the topologist's sine curve cannot be path connected.

5. Given f, g continuous real-valued functions on a metric space (X, ρ), prove that $f + g$, $|f|$, $\max\{f, g\}$, and fg are continuous.

$f + g$ is continuous: Let $x \in X$. Let $\epsilon > 0$ be given. There is δ_f such that for all $a \in X$ with $\rho(x, a) < \delta_f$ implies that $|f(x) - f(a)| < \frac{\epsilon}{2}$. There is a similar δ_g.

Set $\delta = \min\{\delta_f, \delta_g\}$, then for all a with $|x - a| < \delta$, we have that $|(f + g)(x) - (f + g)(a)| = |f(x) + g(x) - (f(a) + g(a))| = |f(x) - f(a) + g(x) - g(a)| \leq |f(x) - f(a)| + |g(x) - g(a)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

$|f|$ is continuous: We will first need a little fact: Let r, s be real numbers. Then $|r| = |r - s + s| \leq |r - s| + |s|$. Thus $|r| - |s| \leq |r - s|$. Similarly $|s| - |r| \leq |s - r| = |r - s|$. Thus $||r| - |s|| \leq |r - s|$.

Let $x \in X$. Let $\epsilon > 0$. There is δ such that for all $a \in X$ with $\rho(x, a) < \delta$ implies that $|f(x) - f(a)| < \epsilon$. Then $||f(x)| - |f(a)|| \leq |f(x) - f(a)| < \epsilon$.

$\max\{f, g\}$ is continuous: $\max\{f(x), g(x)\} = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$ which is continuous by the two problems above.

fg is continuous: Let $x \in X$, and $\epsilon > 0$ be given.

There is δ_1 so that for all $a \in B_{\delta_1}(x), |g(x) - g(a)| < 1$.

There is δ_2 so that for all $a \in B_{\delta_2}(x), |f(x) - f(a)| < \frac{\epsilon}{2(|g(x)| + 1)}$

There is δ_3 so that for all $a \in B_{\delta_3}(x), |g(x) - g(a)| < \frac{\epsilon}{2(|f(x)| + 1)}$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then for all $a \in B_\delta(x)$ we have

$$\begin{aligned}
|f(x)g(x) - f(a)g(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\
&= |f(x)(g(x) - g(a)) + g(a)(f(x) - f(a))| \\
&\leq |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\
&\leq |f(x)||g(x) - g(a)| + (|g(x)| + 1)|f(x) - f(a)| \\
&< |f(x)|\frac{\epsilon}{2(|f(x)| + 1)} + (|g(x)| + 1)\frac{\epsilon}{2(|g(x)| + 1)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

6. Let (X, d_1) and (Y, d_2) be metric spaces. For $(x_1, y_1), (x_2, y_2) \in X \times Y$, define

$$\begin{aligned}\rho_a((x_1, y_1), (x_2, y_2)) &= d_1(x_1, x_2) + d_2(y_1, y_2)\\ \rho_b((x_1, y_1), (x_2, y_2)) &= [d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2]^{1/2}\\ \rho_c((x_1, y_1), (x_2, y_2)) &= \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}\end{aligned}$$

Under each of these metrics, show that the functions $f(x, y) = x$ and $f(x, y) = y$ are continuous from $X \times Y$ to X and $X \times Y$ to Y, respectively.

Proof. By symmetry we assume that $f(x, y) = x$. Also recall the inequality that says for $a, b > 0$, we have

$$\max\{a, b\} \leq [a^2 + b^2]^{1/2} \leq a + b.$$

This follows from the ℓ^p embeddings and the sequence $(a, b, 0, 0, ...) \in \ell^1$.

With this setup, let $(x_1, y_1), (x_2, y_2) \in X \times Y$ such that $\rho_\xi((x_1, y_1), (x_2, y_2)) < \epsilon$ for the desired $\xi = a, b, c$. Then

$$\begin{aligned}d_1(f(x_1, y_1), f(x_2, y_2)) = d_1(x_1, x_2) &\leq \rho_c((x_1, y_1), (x_2, y_2)) \leq \rho_b((x_1, y_1), (x_2, y_2))\\ &\leq \rho_a((x_1, y_1), (x_2, y_2)) < \epsilon.\end{aligned}$$

Therefore f is continuous for each of these metrics on $X \times Y$. □

7. Construct a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is not continuous at $(0, 0)$, but for every straight line L through $(0, 0)$ in $\mathbb{R}^2$, $f|_L$ (the function f restricted to L) is continuous at $(0, 0)$.

Let $g(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$.

Then for $m \in \mathbb{R} \setminus \{0\}$ and $(x, mx) \neq (0, 0)$, we have

$$g(x, mx) = \frac{mx^3}{x^4 + m^2 x^2} = \frac{mx}{x^2 + m^2}$$

to show that $g|L$ is continuous at $(0, 0)$. It is even easier to show for $m = 0$ with the same limit 0 at $(0, 0)$.

Note that for $x \neq 0$, $g(x, x^2) = \dfrac{x^4}{x^4 + x^4} = \dfrac{1}{2}$. $\implies$ g is not continuous at $(0, 0)$. Take $f(x, y) = (g(x, y), 0)$ to get the desired result.

8. Let $C[a, b]$ be the continuous functions on the interval $[a, b] \subset \mathbb{R}$. Suppose that $f, g \in C[a, b]$ with $f(x) < g(x)$ for all $x \in [a, b]$. Is the set $\mathcal{A} = \{h \in C[a, b] \ : \ f(x) < h(x) < g(x) \ \forall \ x \in [a, b]\}$ an open ball in $C[a, b]$?

Solution: We assume the use of $\|\cdot\|_\infty$. First, notice that $\mathcal{A}$ is indeed an open set in $C[a, b]$. (One can argue this just as we argued that the set $B = \{h \in C[a, b] \ : \ h(x) > 0 \text{ on } [a, b]\}$ is open. We did this on Problem Set 2, number 8.)

However, $\mathcal{A}$ need not be an open ball[‡]. Let f and g be given by:

$$g(x) = 1 \text{ (the constant function)}$$

$$f(x) = \begin{cases} x & \text{if } x \in [0, 1/2] \\ \frac{1}{2} & \text{if } x \in [1/2, 1] \end{cases}.$$

If $\mathcal{A}$ were, in this case, an open ball, we would have $\mathcal{A} = B_r(k)$ for some $r > 0$ in $\mathbb{R}$ and $k(x) \in C[0,1]$. Since $g - \epsilon$ and $f + \epsilon$ are in $\mathcal{A}$ for all $0 < \epsilon < \|f - g\|_\infty$, it follows that we must have $r \geq \frac{\|f - g\|_\infty}{2}$. *I.e.*, $r \geq \frac{1}{2}$.

Hence, on $[1/2, 1]$, since $f < k < g$ and $\|k - f\|_\infty < \frac{1}{2}$, we have that $f, g \in B_{r \geq 1/2}(k)$.

Let $\alpha(x) \in C[0,1]$ be defined by $\alpha(x) = \frac{1}{2}$ (the constant function). Then $\alpha(x) \in B_{r \geq 1/2}(k)$, yet $\alpha(x) \notin \mathcal{A}$. So $\mathcal{A} \neq B_{r \geq 1/2}(k)$, which is a contradiction. Thus, for this choice of f and g, $\mathcal{A}$ is not an open ball.

‡ However, $\mathcal{A}$ *might* be an open ball. For example, if $g(x) = f(x) + 1$, then $\mathcal{A}$ is an open ball of radius $\frac{1}{2}$, centered at $f(x) + \frac{1}{2}$.