IMMERSE 2005
Analysis Problem Set 4

1. Let (X,ρ) be a metric space.

(a) For $x, y \in X$, define $\rho_1(x,y) = \dfrac{\rho(x,y)}{1+\rho(x,y)}$. Prove that ρ_1 is also a metric on X.

First, if $\rho_1(x,y) = 0$, then $\rho(x,y) = 0$ implies $x = y$. On the other hand, if $x = y$, then $\rho_1(x,x) = \dfrac{\rho(x,x)}{1+\rho(x,x)} = 0$. And the non-negativity of $\rho(x,y)$ implies the non-negativity of $\rho_1(x,y)$.

The symmetry of $\rho_1(x,y)$ follows from the symmetry of $\rho(x,y)$.

Now, we address the triangle inequality. Let $x, y, z \in X$, and suppose that $\rho(x,y)\rho(x,z)\rho(z,y) \neq 0$. (If any of $\rho(x,y), \rho(x,z)$ or $\rho(z,y)$ is zero, the triangle inequality holds trivially.) Now, suppose that $\rho_1(x,y) > \rho_1(x,z) + \rho_1(z,y)$. Then,

$$\frac{\rho(x,y)}{1+\rho(x,y)} > \frac{\rho(x,z)}{1+\rho(x,z)} + \frac{\rho(z,y)}{1+\rho(z,y)},$$

and multiplying through by $(1+\rho(x,y))(1+\rho(x,z))(1+\rho(z,y))$ (note this term is positive) we have

$$(1+\rho(x,z))(1+\rho(z,y))\rho(x,y) > (1+\rho(x,y))(1+\rho(z,y))\rho(x,z) + (1+\rho(x,y))(1+\rho(x,z))\rho(z,y),$$

which after simplification and canceling like terms yields

$$\rho(x,y) > \rho(x,z) + \rho(z,y) + 2\rho(x,z)\rho(z,y).$$

Since ρ is a metric on X, ρ satisfies the triangle inequality. So we have

$$\rho(x,z) + \rho(z,y) \geq \rho(x,y) > \rho(x,z) + \rho(z,y) + 2\rho(x,z)\rho(z,y),$$

implying $\rho(x,z)\rho(z,y) < 0$, a contradiction. Thus, ρ_2 satisfies the Triangle Inequality.

(b) For $x, y \in X$, define $\rho_2(x,y) = (\rho(x,y))^2$. Prove ρ_2 may or may not be a metric on X.

(i) ρ_2 may not be a metric on X: Let $\rho(x,y) = |x-y|$ be a metric on $\mathbb{R}$. We show that $\rho_2(x,y) = (\rho(x,y))^2 = |x-y|^2$ does not satisfy the Triangle Inequality. Let $x = 0$, $y = \frac{1}{2}$, and $z = \frac{1}{4}$. Then

$$\rho_2(x,y) = \frac{1}{4} > \frac{1}{8} = \frac{1}{16} + \frac{1}{16} = \rho_2(x,z) + \rho_2(z,y).$$

(ii) ρ_2 may be a metric on X: Let $\rho(x,y)$ be the discrete metric on $\mathbb{R}$:

$$\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then $\rho_2(x,y) = \rho(x,y)$, so in this case, ρ_2 is a metric on $\mathbb{R}$.

2. Prove that a subset of a metric space is open if and only if it is a union of open balls.

Solution: Say we are working in the metric space (X,ρ).

$\Rightarrow$ Let $S \subset X$ be open. Then by definition for each x, there is $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \subseteq S$. $x \in B_{\epsilon_x}(x)$, so $S = \bigcup\limits_{x \in S} B_{\epsilon_x}(x)$, a union of open balls.

$\Leftarrow$ Say $S = \bigcup\limits_{\lambda \in \Lambda} B_{\epsilon_\lambda}(s_\lambda)$, a union of open balls. Let $x \in S$. Then there is $\lambda \in \Lambda$ such that $x \in B_{\epsilon_\lambda}(s_\lambda)$. Let $r = \epsilon_\lambda - \rho(s_\lambda, x)$. We will show that $B_r(x) \subseteq B_{\epsilon_\lambda}(s_\lambda) \subseteq S$:

Let $y \in B_r(x)$. Then $\rho(y, s_\lambda) \leq \rho(y, x) + \rho(x, s_\lambda) < r + \rho(x, s_\lambda) = \epsilon_\lambda - \rho(s_\lambda, x) + \rho(x, s_\lambda) = \epsilon_\lambda$. Thus $y \in B_{\epsilon_\lambda}(s_\lambda)$, and we have the desired result. We've shown that every element in S is the center of an open ball contained in S, so by definition, S is open.

3. Let (X, ρ) be a metric space.

a. Suppose that for some $\epsilon > 0$, every open ball of radius ϵ in X has compact closure. Show that X is complete.

Proof. Let $\{x_n\}$ be a Cauchy sequence in X. Then for ϵ given in the statement of the problem, there exists $\underline{N \in \mathbb{N}}$ such that for all $m, n \geq N$ we have $d(x_n, x_m) < \epsilon$. $d(x_N, x_n)$ $\underline{\text{for all } n \geq N}$ $\Longrightarrow$ $\{x_n\}_{n \geq N} \subseteq \overline{B_\epsilon(x_N)}$ which is compact. So there exists a subsequence $x_{n_j} \to x \in \overline{B_\epsilon(x_N)}$.

It is left to show that $\{x_n\}$ converges to x. Let $\delta > 0$ be given. As $\{x_n\}$ is Cauchy, there exists $M \in \mathbb{N}$ such that for all $n, m \geq M$, $d(x_n, x_m) < \delta/2$. Also since $x_{n_j} \to x$, there exist x_{n_j} with $n_j \geq M$ that satisfies $d(x_{n_j}, x) < \delta/2$. Taking $m := n_j$ above, we have for all $n \geq N$, $d(x_n, x) \leq d(x_n, x_{n_j}) + d(x_{n_j}, x) < \delta$.

$\square$

b. Suppose that for each $x \in X$, there is an open $\epsilon > 0$ such that an open ball $B_\epsilon(x)$ has compact closure. Show by means of an example that X need not be complete.

Let $X = (0, \infty)$. Then for each $x \in X$, $B_{x/2}(x)$ has compact closure yet $\{1/n\}$ is a Cauchy sequence in X that does not converge in X.

4. Let (X, ρ) be a metric space. For $x, y \in X$, define $\rho_0(x, y) = \min\{\rho(x, y), 1\}$. Prove that (X, ρ_0) is totally bounded if and only if (X, ρ) is totally bounded.

Proof. First consider when $\epsilon \in (0, 1)$. Then $B_{\epsilon,\rho_0}(x) = B_{\epsilon,\rho}(x)$ for all $x \in X$ $\Longrightarrow$ $\bigcup\limits_{n=1}^{N} B_{\epsilon,\rho_0}(x_n) \supseteq X \iff \bigcup\limits_{n=1}^{N} B_{\epsilon,\rho}(x_n) \supseteq X$.

($\Longrightarrow$) Assume $\epsilon \geq 1$ and (X, ρ_0) is totally bounded. Then there exists a finite set $\{x_n\}_{n=1}^{N}$ such that

$$X \subseteq \bigcup_{n=1}^{N} B_{1/2,\rho_0}(x_n) = \bigcup_{n=1}^{N} B_{1/2,\rho}(x_n) \subseteq \bigcup_{n=1}^{N} B_{\epsilon,\rho}(x_n).$$

($\Longleftarrow$) Assume $\epsilon \geq 1$. Then fix $x \in X$. Then $d_{\rho_0}(x, y) \leq 1$ for all $y \in X$ $\Longrightarrow$ $X \subseteq B_{\epsilon,\rho_0}(x)$. $\square$

5. Let A be a subset of the complete metric space X. Prove that A is totally bounded if and only if the closure of A is compact.

Solution: Set A be a subset of the complete metric space (X, ρ).

[$\Leftarrow$] Suppose that $\overline{A}$ is compact. Then it is sequentially compact, implying $\overline{A}$ is totally bounded. Since $A \subseteq \overline{A}$, A is therefore totally bounded. To see that this is so, note that since $\overline{A}$ is totally bounded, for $\epsilon > 0$ there is a set $\{a_1, \ldots, a_n\} \subseteq \overline{A}$ such that $\bigcup_{i=1}^{n} B_{\frac{\epsilon}{2}}(a_i) \supseteq \overline{A}$. If $a_i \in \overline{A} - A$ for some i, then $B_{\frac{\epsilon}{2}}(a_i) \bigcap A \neq \emptyset$. Say $b_i \in B_{\frac{\epsilon}{2}}(a_i) \bigcap A$ for each i.

Claim: $B_\epsilon(b_i) \supseteq B_{\frac{\epsilon}{2}}(a_i)$ for each $i \in \{1, \ldots, n\}$. *Proof of Claim.* Suppose $\gamma \in B_{\frac{\epsilon}{2}}(a_i)$ for some i. Then

$$\rho(\gamma, b_i) \leq \rho(\gamma, a_i) + \rho(a_i, b_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $\gamma \in B_\epsilon(a_i)$.

Hence, having the claim, we have $A \subseteq \overline{A} \subseteq \bigcup_{i=1}^{n} B_{\frac{\epsilon}{2}}(a_i) \subseteq \bigcup_{i=1}^{n} B_\epsilon(b_i)$, where $b_i \in A$ for each i. Thus, A is totally bounded, as desired.

[$\Rightarrow$] Suppose that A is totally bounded. Now, $\overline{A}$ is closed, and closed subsets of complete metric spaces are complete, so $\overline{A}$ is complete. Moreover, since A is totally bounded, so to is $\overline{A}$. To see this, let $\epsilon > 0$. Since A is totally bounded, there is a set $\{a_1, \ldots, a_n\} \subseteq A$ such that $\bigcup_{i=1}^{n} B_{\frac{\epsilon}{2}}(a_i) \supseteq A$.

Now, suppose $a \in \overline{A}$. Then if $a \notin A$, there is a sequence $\{b_i\}_{i=1}^{\infty} \subseteq A$ such that $\lim_{i \to \infty} b_i = a$. Then there is an $N \in \mathbb{N}$ for which $\rho(b_n, a) < \frac{\epsilon}{2}$ when $n \geq N$. In particular, since $b_N \in A$, $b_N \in B_{\frac{\epsilon}{2}}(a_i)$ for some $i \in \{1, \ldots, n\}$. For such an i, we have

$$\rho(a, a_i) \leq \rho(a, b_N) + \rho(b_N, a_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $a \in B_\epsilon(a_i)$, and we conclude that $\overline{A} \subseteq \bigcup_{i=1}^{n} B_\epsilon(a_i)$. Thus, $\overline{A}$ is totally bounded.

So, since we have shown $\overline{A}$ to be complete and totally bounded, we conclude that $\overline{A}$ is compact, as desired.

6.

(a) Find an example of a decreasing sequence $A_1 \supset A_2 \supset \cdots$ of closed connected subsets of $\mathbb{R}^2$ such that $\cap_{k \geq 1} A_k$ is not connected.

(b) Let $A_1 \supset A_2 \supset \cdots$ be a decreasing sequence of connected compact subsets of $\mathbb{R}^n$. Prove that $\cap_{k \geq 1} A_k$ is connected.

Solution:

(a) Let $I = (-1, 1)$. Let $R_n = I \times (-n, n)$. Let $A_n = \mathbb{R}^2 - R_n$. $R_1 \subset R_2 \subset \cdots$, so $A_1 \supset A_2 \supset \cdots$. R_n is open, so A_n is closed. A_n is $\mathbb{R}^2$ with a finite hole cut in the middle, so given $x, y \in \mathbb{R}^2$, there is a path around the hole that connects them. So A_n is path connected and therefore connected.

On the other hand let

$A := \cap_{k\geq 1} A_k = \{(x,y) \in \mathbb{R}^2 | x \leq -1 \text{ or } x \geq 1\}$
$B := \{(x,y) \in \mathbb{R}^2 | x \leq -1\}$
$C := \{(x,y) \in \mathbb{R}^2 | x \geq 1\}$

Then C, D form a separation of A: $A = C \cup D, \overline{C} \cap D = C \cap D = \emptyset, C \cap \overline{D} = C \cap D = \emptyset$.

(b) Say that $A = \cap_{k\geq 1} A_k$ is disconnected. Then there is a separation B, C. We will show that B and C are closed:

Let $b \in \overline{B}$. $A = B \cup C$. But A is compact, so $A = \overline{A} = \overline{B} \cup \overline{C}$. Thus $b \in A$, so either $b \in B$ or $b \in C$. But $b \in \overline{B}$, and $\overline{B} \cap C = \emptyset$, so $b \in B$. Thus $B = \overline{B}$. Similarly, C is closed as well.

Since B and C are closed sets of a compact subset of $\mathbb{R}^n$, B and C are compact. Let $r = \inf\limits_{b\in B,\, c\in C} d(b,c)$. Certainly r is nonnegative. We show that r is strictly greater than 0:

If $r = 0$ then there is a sequence $\{(b_n, c_n) \in B \times C\}$ such that $\lim\limits_{n\to\infty} d(b_n, c_n) = 0$. But B and C are compact, so there is a subsequence $\{(b'_n, c'_n)\}$ which converges to $(b,c) \in B \times C$. Let $\epsilon > 0$. There is N_0 such that for all $n \geq N_0$, $d(b'_n, c'_n) < \epsilon/3$. There is N_b so that $d(b, b'_n) < \epsilon/3$ for all $n \geq N_b$. There is N_c such that $d(c, c'_n) < \epsilon/3$ for all $n \geq N_c$. Let $N = \max\{N_0, N_b, N_c\}$. Then $d(b,c) \leq d(b, b'_N) + d(b'_N, c'_N) + d(c'_N, c) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$. $d(b,c) < \epsilon$ for all positive ϵ, so $d(b,c) = 0$. Thus $b = c$. But this is a contradiction, since B and C are disjoint.

Now, consider $B' = \bigcup\limits_{b\in B} B_{r/3}(b)$, and $C' = \bigcup\limits_{c\in C} B_{r/3}(c)$.

We claim that $\overline{B'} \cap C' = \emptyset$. Assume $x \in \overline{B'} \cap C'$. Then there are $b \in B$ and $b' \in B'$ with $d(b', x) < r/3$ and $d(b', b) < r/3$. There is also $c \in C$ with $d(x,c) < r/3$. But then $d(b,c) \leq d(b,b') + d(b',x) + d(x,c) < r$, a contradiction by the definition of r.

Similarly, $B' \cap \overline{C'} = \emptyset$

Now define $A'_n = A_n \cap (\mathbb{R}^n - (B \cup C))$. $B \cup C$ is open, so $\mathbb{R}^n - (B \cup C)$ is closed. Thus $A_n \cap (\mathbb{R}^n - (B \cup C))$ is closed and bounded. So A'_n is compact. But $A'_1 \supseteq A'_2 \supseteq \cdots$. If A'_i is nonempty for all i, then $\bigcap\limits_{i=1}^{\infty} A'_i$ is nonempty:

Build a sequence $\{a'_n\}$ by taking a'_i to be some element of A'_i. $A'_1 \supseteq A'_2 \supseteq \cdots$ so $\{a'_n\}$ is in A'_1, which is compact, so there is a convergent subsequence $\{a''_n\}$. Well, since a''_i comes from at least the i^{th} term in $\{a'_n\}$, then for all $j \geq i$, $a''_j \in A'_i$. Thus the tail of $\{a''_n\}$ forms a Cauchy sequence in A'_i. Since A'_i is compact, then l, the limit of $\{a''_n\}$, is in A'_i for all i. Thus l is in the intersection, so $\bigcap\limits_{i=1}^{\infty} A'_i$ is nonempty as desired.

So let $a' \in \bigcap\limits_{i=1}^{\infty} A'_i$. Well, $a' \in A$. But this is a contradiction, since $A = B \cup C \subseteq B' \cup C'$, and A'_i is entirely outside of $B' \cup C'$. Thus there is some empty A'_i. In other words, $A_i \subseteq B' \cup C'$. But consider $A_i \cap B'$, $A_i \cap C'$. $\overline{A_i \cap B'} \cap (A_i \cap C') \subseteq \overline{B'} \cap C' = \emptyset$. Similarly, $(A_i \cap B') \cap \overline{A_i \cap C'} = \emptyset$. Thus if $(A_i \cap B') \neq \emptyset$, and $(A_i \cap C') \neq \emptyset$, then $A_i \cap B'$, $A_i \cap C'$ form a separation of A_i, a contradiction since A_i is connected. Thus we must have $A_i \subseteq B'$ or $A_i \subset C'$. WLOG $A_i \subseteq B'$. Then $A \subseteq B'$, and in particular, $A \cap C = \emptyset$, a contradiction since B, C form a separation of A. $\square$