IMMERSE 2005
Analysis Problem Set 1

2. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. Prove $\lim\limits_{i\to\infty} x_i = x$ if and only if $\lim\limits_{i\to\infty} \|x_i - x\| = 0$.

Solution:

$\Rightarrow$ Suppose $\lim\limits_{i\to\infty} x_i = x$. Let $\epsilon > 0$ be given. We want to find an $N \in \mathbb{N}$ such that for all $n \geq N$, $\|\|x_n - x\| - 0\| < \epsilon$. Well, since $\lim\limits_{i\to\infty} x_i = x$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $\|\|x_n - x\| - 0\| = \|x_n - x\| < \epsilon$. Thus this N suffices.

$\Leftarrow$ Suppose $\lim\limits_{i\to\infty} \|x_i - x\| = 0$. Let $\epsilon > 0$ be given. We want to find an $N \in \mathbb{N}$ such that for all $n \geq N$, $\|x_n - x\| < \epsilon$. Well, since $\lim\limits_{i\to\infty} \|x_i - x\| = 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $\|x_n - x\| = \|\|x_n - x\| - 0\| < \epsilon$. Thus this N suffices.

3. Prove that the space of real numbers $\mathbb{R}$ is complete. *Hint:* Use the Bolzano-Weierstrass Theorem.

Solution: Let $\epsilon > 0$ be given, and let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is Cauchy, we have that there is an $N \in \mathbb{N}$ for which $i \geq j > N$ implies $|x_i| - |x_j| \leq |x_i - x_j| < 1$. So, for each $i > N$, we have $|x_i| < 1 + |x_N|$. Letting $R = \max\{|x_1|, \ldots, |x_{N-1}|, |x_N| + 1\}$, we have $|x_j| \leq R$ for each $j \in \mathbb{N}$; the sequence $\{x_n\}$ is bounded. Furthermore, there is an $N_1 \in \mathbb{N}$ for which $i \geq j > N_1$ implies $|x_i - x_j| < \frac{\epsilon}{2}$.

By the Bolzano-Weierstrass Theorem, we have that there is a convergent subsequence $\{x_{n_i}\} \subseteq \{x_n\}$. Say $\lim_{i\to\infty} x_{n_i} = x$. So, since $\{x_{n_k}\}$ converges to x, we have that there is an $N_2 \in \mathbb{N}$ for which $i > N_2$ implies $|x_{n_i} - x| < \frac{\epsilon}{2}$.

Pick an element x_{n_k} for which $k > N_2$ and $n_k > N_1$. Then, for $i > N_1$, we have

$$\begin{aligned} |x_i - x| &= |x_i - x_{n_k} + x_{n_k} - x| \\ &\leq |x_i - x_{n_k}| + |x_{n_k} - x| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} x_n = x$. Since $\{x_n\}$ was an arbitrary Cauchy sequence in $\mathbb{R}$, and since we have shown that it converges, we have every Cauchy sequence in $\mathbb{R}$ converges; $\mathbb{R}$ is complete.

4. Prove that $\mathbb{R}^n$ is complete, where $n \in \mathbb{N}$.

Solution: Denote $x \in \mathbb{R}^n$ by $x = (x^1, x^2, ..., x^n)$. Note that for $x, y \in \mathbb{R}^n$, we have

$$(x^i - y^i)^2 \leq (x^1 - y^1)^2 + ... + (x^n - y^n)^2 = \|x - y\|^2.$$

To show that $\mathbb{R}^n$ is complete, let $\{x_n\}$ be a Cauchy sequence in $\mathbb{R}^n$. That is, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $j, k > N$, $\|x_j - x_k\| < \epsilon$.

Now for $1 \leq i \leq n$, $|x_j^i - x_k^i| \leq \|x_j - x_k\| < \epsilon$ for all $j, k > N$. Hence each component $\{x_n^i\}$ of $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges in $\mathbb{R}$ by the previous problem. Say these limits are $x^i \in \mathbb{R}$

and $x = (x^1, x^2, ..., x^n) \in \mathbb{R}^n$. Since each component converges, we can say for $\epsilon > 0$ and $1 \leq i \leq n$, there exists N_i such that for all $j > N_i$, $|x_j^i - x^i| < \epsilon/\sqrt{n}$. Therefore

$$\|x_j - x\|^2 = (x_j^1 - x^1)^2 + (x_j^2 - x^2)^2 + ... + (x_j^n - x^n)^2 < \sum_{i=1}^{n} \epsilon^2/n = \epsilon^2$$

for all $j > \max\{N_i : i = 1, ..., n\}$. This can be done for all $\epsilon > 0$, thus $\mathbb{R}^n$ is a complete metric space.

5. Prove that the rationals $\mathbb{Q}$ are not a complete subset of $\mathbb{R}$.

Solution: It suffices to show that there is a sequence of rationals converging to an irrational. We will construct a sequence a rationals converging to $\pi = 3.1415926\dots$ which is known to be irrational:

$$\begin{aligned} x_0 &= 3 \\ x_1 &= 3.1 \\ x_2 &= 3.14 \\ x_3 &= 3.141 \\ x_4 &= 3.1415 \\ x_5 &= 3.14159 \\ &\vdots \end{aligned}$$

6. Let A be a subset of $\mathbb{R}^n$. Prove that the closure of A contains all of its limit points, or in other words, $\overline{\overline{A}} = \overline{A}$.

Solution: Since it is clear that $\overline{A} \subseteq \overline{\overline{A}}$, it suffices to show that $\overline{\overline{A}} \subseteq \overline{A}$.

Let $a \in \overline{\overline{A}}$. Then $B\left(a, \frac{1}{2n}\right)$ intersects $\overline{A}$ in at least one point, say b_n. Similarly, since $b_n \in \overline{A}$, we know that $B\left(b_n, \frac{1}{2n}\right)$ intersects A in at least one point, say a_n. This holds for each $n \in \mathbb{N}$. Consider the sequence formed by these a_n. We have $\|a - a_n\| \leq \|a - b_n\| + \|b_n - a_n\| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$, by the triangle inequality. So, for $\epsilon > 0$, there is an $N(\epsilon) \in \mathbb{N}$ for which $\frac{1}{n} < \epsilon$ when $n > N$. Therefore $n > N$ implies $\|a - a_n\| \leq \frac{1}{n} < \epsilon$, implying a is a limit point of $\{a_n\} \subseteq A$. Thus, $a \in \overline{A}$, which was to be shown.

7. If A is a bounded subset of $\mathbb{R}$ show that sup A and inf A are elements of $\bar{A}$.

Solution: Since A is bounded, the supremum exists. Define $x =$ sup A. Attempting a contradiction, suppose $x \in \mathbb{R} \backslash \bar{A}$. Since $\bar{A}$ is closed its complement is open. So, $\exists\, B_r(x) \subset \mathbb{R} \backslash \bar{A}$. Notice that $x - \frac{r}{2} \in B_r(x)$ and $B_r(x) \bigcap \bar{A} = \emptyset$ so $\forall a \in A$, $a < x - \frac{r}{2} < x$. This contradicts that x is the supremum. For the infemum, imitate the proof considering the element $x + \frac{r}{2}$.

8. Let $S \subset [0, \infty)$ such that $u = \sup S, u < 1$. Additionally, assume that if $x, y \in S$, and $x < y$, then $\frac{x}{y} \in S$. Show $u \in S$.

Solution: Assume $u \notin S$. Since $u = \sup S$, S must be infinite. Thus, there exists an infinite, strictly increasing sequence $\{x_i\}_{i=0}^{\infty} \subseteq S$ such that the limit of $\{x_i\}_{i=0}^{\infty}$ is u. Consider $\{y_i\}_{i=0}^{\infty}$ where $y_i = x_i/x_{i+1}$. Since $x_i \in S$ and $x_i < x_{i+1}$, then we have that $\{y_i\}_{i=0}^{\infty} \subseteq S$. But since $\{x_i\}_{i=0}^{\infty}$ converges, $\{y_i\}_{i=0}^{\infty}$ must converge to 1. Thus, $u \geq 1$. This contradicts the fact that $u < 1$, so our original assumption that $u \notin S$ is false. Thus, $u \in S$.

9. First, show $\mathbb{R} \backslash \mathbb{Q}$ is dense in $\mathbb{R}$. Then show int$(\mathbb{Q}) = \emptyset$.

Solution: Let $x \in \mathbb{R}$. If $x \in \mathbb{R} \backslash \mathbb{Q}$, then $x \in \overline{\mathbb{R} \backslash \mathbb{Q}}$, , so assume $x \in \mathbb{Q}$. Then the sequence $\{\frac{\pi}{n} + x\}_{n=1}^{\infty}$ converges to x, and each element of the sequence is irrational. Hence, x is a limit point of $\mathbb{R} \backslash \mathbb{Q}$, so

$x \in \overline{\mathbb{R}\backslash\mathbb{Q}}$. Therefore, $\mathbb{R} \subseteq \overline{\mathbb{R}\backslash\mathbb{Q}}$, and so $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$.

Next show $\text{int}(\mathbb{Q}) = \emptyset$.

Solution: Assume to the contrary that $\text{int}(\mathbb{Q}) \neq \emptyset$. Then $\exists\ x \in \text{int}(\mathbb{Q})$. Now, $\text{int}(\mathbb{Q})$ is an open set, so $\exists\ r > 0$ such that $B_r(x) \subseteq \text{int}(\mathbb{Q})$. But $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$ so it is dense in the subset $\text{int}(\mathbb{Q})$. So $\exists\ z \in \mathbb{R}\backslash\mathbb{Q}$ such that $z \in B_r(x)$. But $\text{int}(\mathbb{Q})$ cannot contain any irrational numbers. Hence, $\text{int}(\mathbb{Q}) = \emptyset$.

10. Let A be a subset of $\mathbb{R}^n$. Prove that if A is compact, then A is closed and bounded.

Solution: (Compact implies bounded) Suppose not. Suppose A is not bounded.

Note: We are going to construct a sequence in A that has no convergent subsequence.

Let $x \in A$. Then we know there is a $y_1 \in A$ such that $y_1 \notin B_1(x)$. Continue this process for all $i \in \mathbb{N}$. That is, construct $y_i \in A$ such that $y_i \notin B_i(x)$. Clearly $\{y_1\}_{i=1}^{\infty}$ is a sequence in A that has no convergent subsequence. But this is a contradiction because A is compact. Hence A must be bounded.

(Compact implies closed)

Toward a contradiction, assume A is not closed. Then there exists a sequence $\{a_i\} \to a$ such that $a \notin A$. Every subsequence $\{a_{i_k}\} \to a$ which violates our definition of compact. Contradiction! Therefore, A must be closed.

11. Let A, B be subsets of $\mathbb{R}^n$.
We define the sum $A + B = \{x \in \mathbb{R}^n : x = a + b, a \in A, b \in B\}$.

- **(a) Show that the sum of a closed set in $\mathbb{R}^n$ and a compact set in $\mathbb{R}^n$ is closed.**
- **(b) Is the sum of two closed sets in $\mathbb{R}^n$ closed? (Prove, or find a counter example.)**

Solution:

(a) Consider $A + B$, where A is closed and B compact. Let x be a limit point of $A + B$. So there exists a sequence $\{x_i\} \subseteq A + B$ such that $\lim_{i\to\infty} x_i = x$, where $x_i = a_i + b_i$. Since B is compact, the sequence $\{b_i\}$ has a subsequence converging in B: $\{b_{i_k}\} \to b \in B$.

So consider $x_{i_k} = a_{i_k} + b_{i_k} \to x$, and look at $x_{i_k} - b_{i_k} = a_{i_k}$. Since $x_{i_k} \to x$ and $b_{i_k} \to b$, $\lim_{k\to\infty}(x_{i_k} - b_{i_k}) = x - b$, implying $\lim_{k\to\infty} a_{i_k} = x - b$. Since A is closed, we must have $x - b \in A$. So x can be written as $x = a + b$ for some $a \in A$ and $b \in B$, implying $x \in A + B$. Thus, $A + B$ is closed.

(b) NO. A counter example is as follows:

Let $A = \{-n\}_{n=1}^{\infty}$ and $B = \left\{n + \frac{1}{n}\right\}_{n=2}^{\infty}$.

Clearly, A and B are closed sets, but they are not bounded.

Now, we note that $\left\{\frac{1}{n}\right\}_{n=2}^{\infty}$ is a subsequence of the sum $A + B$.

Therefore, since $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and 0 is not contained in the sum $A + B$, the sum $A + B$ is not closed, as needed.