Problem Set 8

IMMERSE 2005 Analysis Course

1. The Haar scaling function ϕ is given by:

$$\phi(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Prove that $\{\phi(x-k) \,:\, k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$.

   (b) Prove that $\left\{2^{\frac{j}{2}}\phi(2^j x) \,:\, j \in \mathbb{Z}\right\}$ is *not* an orthonormal collection.

2. The Haar wavelet function ψ is given by:

$$\psi(x) = \begin{cases} 1 & 0 \leq x < \frac{1}{2} \\ -1 & \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Prove that the collection

$$\left\{2^{\frac{j}{2}}\psi(2^j x - k) \,:\, j,k \in \mathbb{Z}\right\}$$

   is an orthonormal collection in $L^2(\mathbb{R})$. Use the following steps to help.

   (a) Prove the result for the cases when $j = 0$. Do the same when restricted to $k = 0$.

   (b) Determine the interval where the function $2^{\frac{j}{2}}\psi(2^j x - k)$ is nonzero.

   (c) Prove the result for arbitrary $j, k \in \mathbb{Z}$.

3. Given functions $f, g \in Ł^2(\mathbb{R})$, the *convolution* of f and g is defined to be the function $f * g$ given by :

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$$

   Given ϕ the Haar scaling function defined in exercise #1, compute and sketch the graph of $\phi * \phi$.

4. Prove that $\phi * \phi$ is a refinable function. (Recall from class that ϕ is refinable.) Find the refinement equation for $\phi * \phi$.