Problem Set 6

IMMERSE 2005 Analysis Course

1. We know that if $(V, \|\cdot\|_V)$ is a normed vector space (over $\mathbb{R}$ or $\mathbb{C}$), then V is also a metric space with metric $\rho(u,v) = \|u-v\|_V$. Given a metric space (X,d), the natural way to construct a norm would be to define $\|x\|_X = d(x,0)$ for $x \in X$. However, as you've seen in previous homework, this may not actually define a norm on X even if X is a vector space.

   Prove that if X is a vector space and d is a metric on X such that

   - $d(cx,0) = |c|\, d(x,0)$ for all scalars c and all $x \in X$
   - $d(x,y) = d(x-y,0)$ for all $x, y \in X$

   then $\|x\|_X = d(x,0)$ defines a norm on X.

2. Let $f_n(x) = xne^{-nx}$ for all $x \geq 0$ and $n \geq 1$. Show that $\{f_n\}$ converges to zero on $[0,\infty)$ pointwise but not uniformly.

3. For $n \in \mathbb{N}$, define $f_n(x) = \dfrac{\tan^{-1}(x)}{n}$. Show that $\{f_n\}$ converges uniformly on $\mathbb{R}$.

4. Prove Dini's Theorem: suppose that f and f_n are continuous functions on $[a,b]$ such that for each $n \in \mathbb{N}$, $f_n(x) \leq f_{n+1}(x)$ for all $x \in [a,b]$. If $f_n \to f$ pointwise, then $f_n \to f$ uniformly.

5. Give an example of a sequence of discontinuous functions f_n that converge uniformly to a continuous function.

6. (a) Let (X,ρ) and (Y,d) be metric spaces, $f_n : X \to Y$ $(n \in \mathbb{N})$, and $f : X \to Y$. Prove that if f_n is continuous for each $n \in \mathbb{N}$ and $f_n \to f$ uniformly on X, then f is continuous on X.

   (b) Use the above fact to prove that $(C[a,b], \|\cdot\|_\infty)$ is complete.

7. Suppose $f_k : D \subset \mathbb{R} \to \mathbb{R}$. We say $\sum_{i=1}^{\infty} f_i(x)$ is uniformly Cauchy on D if for every $\epsilon > 0$ there is an N so that

   $$\left\| \sum_{i=k+1}^{l} f_i(x) \right\| \leq \epsilon$$

   whenever $x \in D$ and $l > k \geq N$. Prove that $\sum_{i=1}^{\infty} f_i(x)$ converges uniformly if and only if it is uniformly Cauchy.

8. Let x_k be the sequence of real numbers such that the kth element of the sequence is $\frac{1}{k}$ and all other elements are zero. Clearly for each $k \in \mathbb{N}$, $x_k \in l^p$ for $1 \leq p \leq \infty$. Is $\{x_k\}_{k=1}^{\infty}$ summable in l^∞? in l^2? in l^1?

9. For each $n \in \mathbb{N}$, let $f_n(x) = \dfrac{x^2}{n(1+x^2)}$. For which values of $a < b$ is $\sum_{n=1}^{\infty} f_n(x)$ summable in $(C[a,b], \|\cdot\|_\infty)$?