

Problem Set 5
IMMERSE 2005 Analysis Course

1. Prove that the three characterizations given in class of a continuous function are equivalent.
2. Let (X, ρ) be a metric space and let $(V, \|\cdot\|)$ be a normed vector space. Let $C_b(X, V)$ be the vector space of all bounded continuous functions $f : X \rightarrow V$. Prove that

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}$$

is a norm on $C_b(X, V)$.

3. For each of the following statements, prove the statement or find a counterexample which disproves it. In each part, f is a continuous function from a metric space (X, ρ) to a metric space (Y, σ) .
 - (a) If C is compact in X , then $f(C)$ is compact in Y .
 - (b) If C is connected in X , then $f(C)$ is connected in Y .
 - (c) If C is path connected in X , then $f(C)$ is path connected in Y .
4. Let $A \subset X$ be path connected. Is \overline{A} necessarily path connected?
5. Given f, g continuous real-valued functions on a metric space (X, ρ) , prove that $f+g, |f|, \max\{f, g\}$, and fg are continuous.
6. Let (X, d_1) and (Y, d_2) be metric spaces. For $(x_1, y_1), (x_2, y_2) \in X \times Y$, define

$$\begin{aligned}\rho_a((x_1, y_1), (x_2, y_2)) &= d_1(x_1, x_2) + d_2(y_1, y_2) \\ \rho_b((x_1, y_1), (x_2, y_2)) &= [d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2]^{\frac{1}{2}} \\ \rho_c((x_1, y_1), (x_2, y_2)) &= \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}\end{aligned}$$

Under each of these metrics, show that the functions $f(x, y) = x$ and $f(x, y) = y$ are continuous from $X \times Y$ to X and $X \times Y$ to Y , respectively.

7. Construct a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that f is not continuous at $(0, 0)$, but for every straight line L through $(0, 0)$ in \mathbb{R}^2 , $f|_L$ (the function f restricted to L) is continuous at $(0, 0)$.
8. Let $C[a, b]$ be the continuous functions on the interval $[a, b] \subset \mathbb{R}$. Suppose $f, g \in C[a, b]$ with $f(x) < g(x)$ for all $x \in [a, b]$. Is the set $\{h \in C[a, b] : f(x) < h(x) < g(x) \forall x \in [a, b]\}$ an open ball in $C[a, b]$?