

Problem Set 10
IMMERSE 2005 Analysis Course

1. (a) Prove Cantor's Intersection Theorem: a decreasing sequence of nonempty compact subsets $A_1 \supset A_2 \supset \dots$ of a metric space (X, ρ) has nonempty intersection.
- (b) Let $X \subset \mathbb{R}^2$ be closed, $K(X)$ be the set of all compact subsets of X , and d_H be the Hausdorff metric. Prove $(K(X), d_H)$ is complete. Hint: for $\{A_n\}$ Cauchy in $K(X)$, consider

$$A = \bigcap_{k \geq 1} \overline{\bigcup_{i \geq k} A_i}$$

2. Find an IFS on \mathbb{R} that generates the Cantor set.
3. Consider the eight maps on \mathbb{R}^2 given by $T_i x = Ax + \frac{1}{3}B_i$ for $i = 1, \dots, 8$, where

$$A = \frac{1}{3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, B_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, B_6 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B_7 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \text{ and } B_8 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

- (a) Let S be the square with vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Graph TS , T^2S , and T^3S (T defined in the usual way). The fixed point of T is called the Sierpinski Carpet.
- (b) (optional) If you have experience with programming, write a program to generate an approximation to the Sierpinski Carpet.
4. Consider the four maps on \mathbb{R}^2 given by $T_i x = Ax + B_i$ for $i = 1, 2, 3, 4$, where

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ and } B_4 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

- (a) Show that the fixed points of the T_i 's form the vertices of a square S .
- (b) Compute TS and T^2S (T defined in the usual way).
- (c) If you calculated $T^n S$ for a few more values of n , then you would see that the fixed point of T is the octagon O pictured below. Calculate $T_i O$ for each $i = 1, 2, 3, 4$ to see how O is covered by the $T_i O$. Notice that unlike all our examples from class, here there is a great deal of overlap between the $T_i O$.

