Problem Set 1

IMMERSE 2005 Analysis Course

1. Read the Beer paper. Make a list of the topics you think we will need to learn about in order to understand this paper well. What are the words we will need to define? (We will be collecting this list, so please actually write it up and hand it in on Tuesday.)

2. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $\mathbb{R}^n$. Prove $\lim_{i\to\infty} x_i = x$ if and only if $\lim_{i\to\infty} \|x_i - x\| = 0$.

3. Prove that the space of real numbers $\mathbb{R}$ is complete. *Hint:* Use the Bolzano-Weierstrass Theorem.

4. Prove that $\mathbb{R}^n$ is complete, where $n \in \mathbb{N}$.

5. Prove that the rationals $\mathbb{Q}$ are not a complete subset of $\mathbb{R}$.

6. Let A be a subset of $\mathbb{R}^n$. Prove that the closure of A contains all its limit points, or in other words, $\overline{\overline{A}} = \overline{A}$.

7. Recall if A is a subset of $\mathbb{R}$, we define the *supremum* of A, if it exists, to be the smallest number $s \in \mathbb{R}$ such that $s \geq a$ for all $a \in A$. This is denoted $\sup A = s$. Similarly, the *infemum* of A, if it exists, is the largest number $i \in \mathbb{R}$ such that $i \leq a$ for all $a \in A$, and is denoted $\inf A = i$.

   If A is a bounded subset of $\mathbb{R}$, show that $\sup A$ and $\inf A$ are elements of $\overline{A}$.

8. Let $S \subset [0, \infty)$ such that $u = \sup S$, $u < 1$. Additionally, assume that if $x, y \in S$ and $x < y$, then $\frac{x}{y} \in S$. Show $u \in S$.

9. We define the *interior* of a set A to be the largest open set contained in A. We say that a set A is *dense* in the set B if $B \subseteq \overline{A}$.

   (a) Show that the set of irrational real numbers is dense in $\mathbb{R}$.

   (b) Show that the set of rational real numbers $\mathbb{Q}$ has empty interior.

10. Let A be a subset of $\mathbb{R}^n$. Prove if A is compact, then A is closed and bounded.

11. Let A, B be subsets of $\mathbb{R}^n$. We define the sum $A + B = \{x \in \mathbb{R}^n : x = a + b,\ a \in A, b \in B\}$.

    (a) Show that the sum of a closed set in $\mathbb{R}^n$ and a compact set in $\mathbb{R}^n$ is closed.

    (b) Is the sum of two closed sets in $\mathbb{R}^n$ closed? (Prove, or find a counterexample.)