

Dissections of $p : q$ rectangles into thirteen $p : q$ rectangular elements

Charles H. Jepsen*, Ming Gu

Department of Mathematics, Grinnell College, Grinnell, Iowa 50112

Abstract

We determine all simple perfect dissections of $p : q$ rectangles into thirteen $p : q$ rectangular elements. A computer search shows there are 26 such dissections. Previous work yielded only 8 such dissections into at most twelve rectangular elements.

Key words: Simple perfect dissection; c -net; p -net

1 Introduction

We consider the problem of dissecting a $p : q$ rectangle into smaller $p : q$ rectangles. Here p and q are relatively prime integers and $p : q$ denotes the ratio of height to width of a rectangle. (The well-known problem of squaring the square is the special case where $p = q = 1$.) The smaller rectangles are the *elements* of the dissection; the number of elements is the *order* of the dissection. The condition imposed that makes the problem challenging is that the dissections are *perfect*, i.e., no two elements have the same size. We are also interested in dissections that are *simple*, i.e., no proper subset of elements forms a rectangle. These conditions imply that the elements fit together in a nonsymmetric way as illustrated by Figures 2 through 5 below.

* Corresponding author. Tel.: 641-269-4203; fax: 641-269-4984.

Email address: jepsen@math.grinnell.edu, gu@grinnell.edu (Charles H. Jepsen*, Ming Gu).

In [4] we found all simple perfect dissections of $p : q$ rectangles into at most twelve $p : q$ rectangles. There are two dissections of order 10 (each having ratio $1 : 2$), three of order 11 (with ratios $1 : 3$, $1 : 5$, $1 : 7$), and three of order 12 (ratios $4 : 7$, $3 : 7$, $1 : 9$). Previously, in [3] we had investigated dissections of $1 : 2$ rectangles.

In this paper, we find all simple perfect dissections of $p : q$ rectangles into thirteen $p : q$ rectangles. The number of dissections of order 13 is 26, a sharp increase over the 8 dissections of order at most 12. The ratios range from $8 : 9$ to $1 : 7$. See Table 1 for the complete list. The code is analogous to the Bouwkamp code for dissections into squares. Each number is the shorter side of an element. A number with a prime denotes an element with horizontal orientation; no prime means vertical orientation. Figures 2 through 5 below show four of these 26 rectangles. Note the interesting nature of the large-sized $3 : 4$ rectangle in Figure 3. With the dissected rectangle oriented horizontally, 11 of the 13 elements have vertical orientation. Note also in Figure 5 that with the $1 : 7$ rectangle scaled to fit on a page, the smallest element essentially disappears.

In §2, we discuss the theory used to produce simple dissections, and in §3 we describe the computation scheme that yielded the dissections.

2 Theory of the Dissections

Three steps are involved in finding a simple dissection of a $p : q$ rectangle into n $p : q$ rectangular elements.

- 1) Find all possible arrangements of n smaller rectangles that form a simple dissection of a larger rectangle. According to the theory developed in [2], each such arrangement comes from a p -net with n edges which in turn comes from a c -net (a 3-connected planar graph) with $n + 1$ edges.
- 2) For each p -net with n edges, find the possible ratios $p : q$ and orientations of the n $p : q$ rectangles that form a dissection of a $p : q$ rectangle.
- 3) Given the ratio $p : q$ and the orientations of the n $p : q$ rectangles, find the heights and widths of the rectangles.

We describe each step, using the approach and notation in [3] and [4]. The first step contains a new feature.

- 1) The starting point is a list of all c -nets with $n + 1$ edges. (Drawings of c -nets with $n \leq 13$ can be found in [1].) Given a c -net with m vertices labeled

$1, 2, \dots, m$ in some order, denote an edge joining vertices i and j , $i < j$, by (ij) . For each edge (ij) , the c -net yields a p -net obtained by deleting that edge and relabeling vertices by interchanging 1 and i , j and m . See Figure 1 for an example.

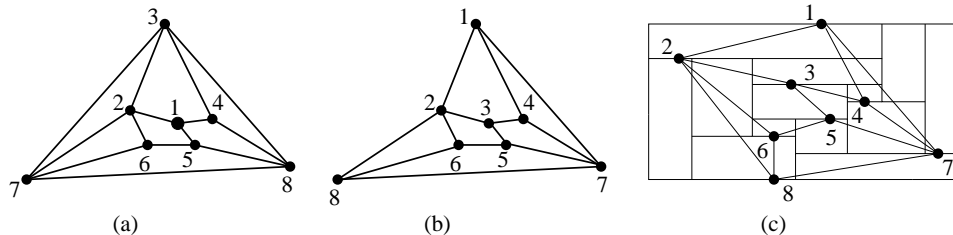


Fig. 1. A c -net yields a simple dissection.

(a) A c -net with 14 edges and 8 vertices.

(b) The p -net obtained by removing the edge joining vertices 3 and 7.

(c) A dissection given by the p -net.

For $n \leq 12$, the numbers were small enough that the p -nets were generated by hand from the list of c -nets. This becomes too tedious for $n = 13$. An algorithm for computer generation follows from the following elementary property of matrix multiplication.

Note Let σ be a permutation of $\{1, 2, \dots, m\}$ and let P be the permutation matrix obtained from the $m \times m$ identity matrix by rewriting its rows in the order $\sigma(1), \sigma(2), \dots, \sigma(m)$. If S is an $m \times m$ matrix whose (k, l) entry is s_{kl} , then the (k, l) entry in the matrix $P^T S P$ is $s_{\sigma(k)\sigma(l)}$.

Consequence Let G be a graph with m vertices and let A be the adjacency matrix of G with its vertices labeled $1, 2, \dots, m$ in some order. For σ as in the note, let G' be the graph obtained by relabeling the vertices in G as $\sigma(1), \sigma(2), \dots, \sigma(m)$. Then with P as in the note, $P^T A P$ is the adjacency matrix of G' .

Given a c -net with vertices $1, 2, \dots, m$, we get the p -net corresponding to edge (ij) as follows. Let A be the adjacency matrix for the c -net and let P be the matrix obtained from the $m \times m$ identity matrix by interchanging rows 1 and i , j and m . Note that $P^T = P$. Form matrix $P A P$ and then change its entries $(1, m)$ and $(m, 1)$ from 1 to 0. The result is the adjacency matrix for the desired p -net.

2) A p -net with m vertices $1, 2, \dots, m$ and n edges (ij) yields a dissection of a rectangle as illustrated in Figure 1 (c). We pay particular attention to the horizontal lines through the vertices and form a system of equations by equating the sum of the widths of the elements above and below a line. The system of equations can be expressed in matrix form as

$$B_1 \vec{w}_1 = \vec{0}, \quad (1)$$

where B_1 is the $(m-1) \times (n+1)$ matrix and \vec{w}_1 is the column $(n+1)$ -vector obtained as follows. Say edge (ij) in the p -net yields element E_{ij} in the dissection with width w_{ij} and height h_{ij} and let w and h denote the width and height of the dissected rectangle. Let \vec{w}_1 be the column vector whose first n entries are w_{ij} and whose last entry is w . (Similarly for the vector \vec{h}_1 below.) Let B be the $(m-1) \times n$ matrix whose columns are labeled by the (ij) and where:

- the entry in the i th row of column (ij) is 1;
- the entry in the j th row of column (ij) is -1 whenever $j < m$;
- all other entries are 0.

Then B_1 is the matrix whose first n columns are the columns of B and whose last column has first entry -1 and all other entries 0. In effect, matrix B_1 encodes the information contained in the p -net.

We now encode the possible orientations of the n rectangular elements in 2^n diagonal matrices whose diagonal entries are either x or $1/x$. Denote any such matrix by $C(x)$. Form an $(n+1) \times (n+1)$ diagonal matrix $C_1(x)$ whose first n diagonal entries are those in $C(x)$ and whose last diagonal entry is $-x$. Then

$$C_1(x) \vec{h}_1 = \vec{w}_1. \quad (2)$$

Now introduce new variables y_1, \dots, y_{m-1} where y_i is the distance between the horizontal lines through vertex i and vertex m , and let \vec{y} be the column $(m-1)$ -vector of the y_i . Then $h_{ij} = y_i - y_j$ says

$$\vec{h}_1 = B_1^T \vec{y}. \quad (3)$$

Putting equations (1) – (3) together, we get

$$(B_1 C_1(x) B_1^T) \vec{y} = \vec{0}. \quad (4)$$

If we replace matrix $B_1 C_1(x) B_1^T$ by $B_1 (x C_1(x)) B_1^T$, then its determinant $\det[B_1 (x C_1(x)) B_1^T]$ is a polynomial in x . It follows from (4) that:

If $p : q$ is a ratio for a simple dissection of a $p : q$ rectangle into $p : q$ rectangular elements, then either p/q or q/p is a rational zero of the polynomial $\det[B_1 (x C_1(x)) B_1^T]$.

3) Given a rational zero $x = p/q$ of the polynomial $\det[B_1(xC_1(x))B_1^T]$, we have both a possible ratio $p : q$ for a dissection and the orientation of the n rectangular elements (the values of x and $1/x$ on the diagonal of $C(x)$). Now use a nonzero rational solution \vec{y} of (4) to find \vec{h}_1 from (3) and \vec{w}_1 from (2), thus finding the heights and widths of the elements.

For instance, in the example in Figure 1, we have

$$B_1 = \begin{matrix} & (12)(14)(17)(23)(26)(28)(34)(35)(47)(56)(57)(68)(78) \\ \left[\begin{array}{cccccccccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 \end{array} \right. \end{matrix}$$

and if we take

$C_1(x) = \text{diag} [x, x, x, 1/x, x, x, x, x, x, x, x, x, 1/x, -x]$, then

$$B_1(xC_1(x))B_1^T = \begin{bmatrix} 2x^2 & -x^2 & 0 & -x^2 & 0 & 0 & -x^2 \\ -x^2 & 3x^2 + 1 & -1 & 0 & 0 & -x^2 & 0 \\ 0 & -1 & 2x^2 + 1 & -x^2 & -x^2 & 0 & 0 \\ -x^2 & 0 & -x^2 & 3x^2 & 0 & 0 & -x^2 \\ 0 & 0 & -x^2 & 0 & 3x^2 & -x^2 & -x^2 \\ 0 & -x^2 & 0 & 0 & -x^2 & 3x^2 & 0 \\ -x^2 & 0 & 0 & -x^2 & -x^2 & 0 & 3x^2 + 1 \end{bmatrix}$$

and $\det[B_1(xC_1(x))B_1^T] = -26x^{10}(x-2)(x+2)(x^2+1)$.

With $x = 2$, $\vec{y} = [60, 28, 40, 48, 35, 21, 44]^T$,

$$\vec{h}_1 = [32, 12, 16, -12, 7, 28, -8, 5, 4, 14, -9, 21, 44, 60]^T,$$

$$\vec{w}_1 = [64, 24, 32, -6, 14, 56, -16, 10, 8, 28, -18, 42, 22, 120]^T.$$

For certain elements E_{ij} , we see that h_{ij} and w_{ij} are negative. This indicates an alternative placement for that element. For instance, element E_{23} lies above the bottom edge of E_{12} instead of below it. This example is the dissection shown in Figure 4 below.

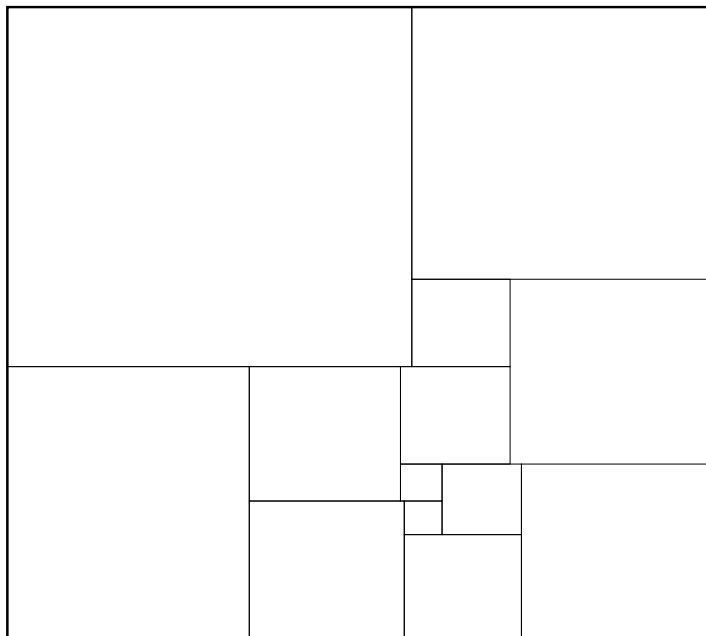


Fig. 2. Dissection of a 1504×1692 rectangle (ratio 8 : 9).
 $(856', 648')$ $(208', 440')$ $(576, 320', 232')$ $(88', 168', 416')$ $(328', 80')$ $(248')$

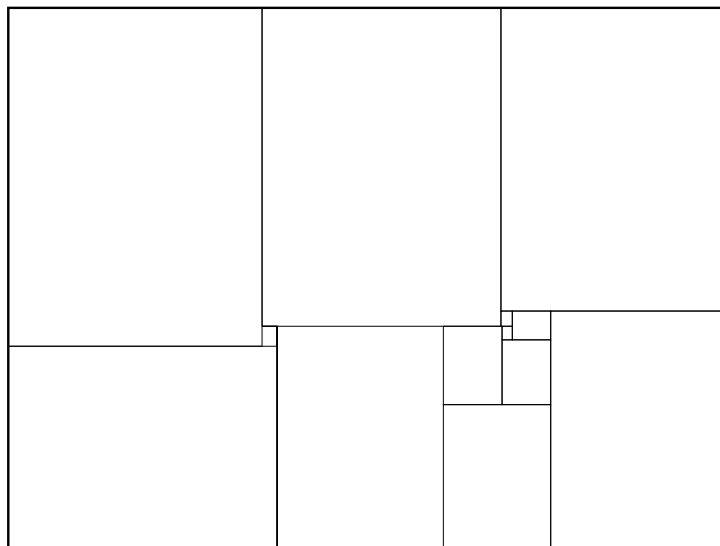


Fig. 3. Dissection of a 2700×3600 rectangle (ratio 3 : 4).
 $(1269, 1194, 1137)$ $(57, 144', 888)$ $(75, 831, 294, 51)$ (243) $(1008')$ (537)

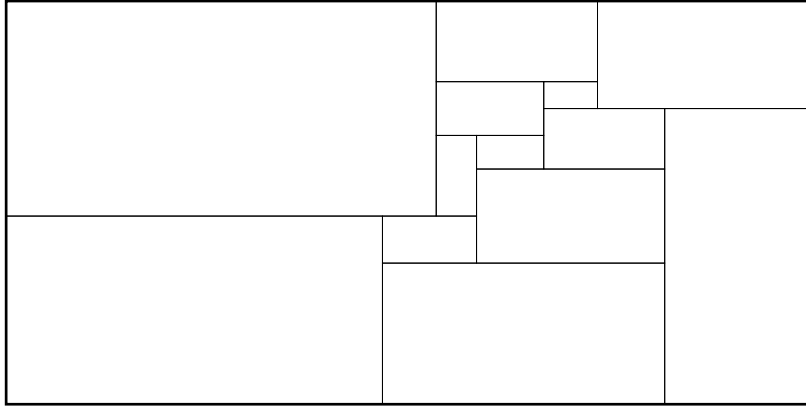


Fig. 4. Dissection of a 60×120 rectangle (ratio 1 : 2).
 $(32', 12', 16')$ $(8', 4')$ $(9', 22)$ $(6, 5')$ $(14')$ $(28', 7')$ $(21')$

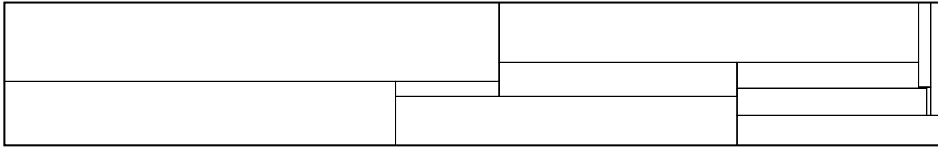


Fig. 5. Dissection of a 248×1736 rectangle (ratio 1 : 7).
 $(137', 104', 21, 28)$ $(59', 45')$ $(111', 26')$ $(2', 7)$ $(47')$ $(85')$ $(52')$

3 Computation of the Dissections

We now describe the computer programs used to find all simple dissections of order 13. We used Maple (version 6.01) for the computation. (The programming was done by the second author as a summer research project under the direction of the first author.) To obtain copies of these programs, see the Web site http://www.math.grinnell.edu/~jepsen/dissect_into_13/.

Three programs were used, one for each of the three steps explained in the preceding section. In the first program, the input consists of data describing a c -net and the output is the matrix B_1 for each of the p -nets corresponding to the c -net. Since there are 37 c -nets with 14 edges, we get $37 \cdot 14 = 518$ files containing matrices B_1 . The second program inputs a matrix B_1 , uses a grey code to create 2^n diagonal matrices $C_1(x)$, computes the $m \times m$ determinant $d(x) = \det[B_1(xC_1(x))B_1^T]$, and finds the zeros of the polynomial $d(x)$. A positive rational zero other than 1 and the diagonal entries of $C(x)$ are written to an output file. Finally, a third program reads this output file, finds the heights and widths of the rectangular elements, and filters out results that contain elements of size 0 or contain elements of the same size (thus eliminating dissections that are not perfect). All answers were checked by hand, and the final list contained the 26 dissections in Table 1.

The programs were run on a Pentium III workstation with a clockspeed of 700 megahertz and 128 megabytes of RAM. After considerable streamlining and experimentation, the running time for each p -net was trimmed to two hours (the second program consuming almost all of this time). Also, to overcome Maple's consumption of memory, a shell script was written in which Maple was entered, a p -net was run, Maple was exited, and the process was repeated. The computation was completed within a five-day period using 12 workstations.

We are convinced that a considerable increase in computing power will be needed to complete the cases $n = 14$ and beyond.

Table 1. Simple perfect dissections of $p : q$ rectangles of order 13

Ratio Dimensions	Code
8 : 9 1504 × 1692	(856', 648') (208', 440') (576, 320', 232') (88', 168', 416') (328', 80') (248')
7 : 8 847 × 968	(504', 392) (140', 203') (343', 161') (77', 63') (70', 196') (182', 56') (126')
7 : 9 896 × 1152	(553', 441) (343', 119', 63', 28') (14', 329') (42') (56', 7') (49') (224')
7 : 9 1316 × 1692	(553', 385', 378') (7', 371') (392') (329', 224') (28', 441) (105', 539') (434')
3 : 4 2700 × 3600	(1269, 1194, 1137) (57, 144', 888) (75, 831, 294, 51) (243) (1008') (537)
2 : 3 150 × 225	(78', 56, 52) (72', 6', 4, 32') (28', 18') (10', 8') (40') (38')
2 : 3 200 × 300	(96, 74', 62') (12', 50') (48', 38') (88') (20, 52) (56', 8') (32)
2 : 3 600 × 900	(332', 192', 114) (14, 100) (144', 86) (186) (268', 64') (40, 176) (136)
3 : 5 276 × 440	(141', 81', 54') (27', 45) (60', 48') (75') (135', 66') (3', 72') (69')
4 : 7 2144 × 3752	(896, 820', 812') (276', 536') (552', 268') (284', 260') (796') (112, 772') (576')
1 : 2 60 × 120	(32', 12', 16') (8', 4') (9', 22) (6, 5') (14') (28', 7') (21')
1 : 2 80 × 160	(42', 29, 17, 30) (5, 6') (2, 10) (38', 8) (7) (22') (20')
1 : 2 83 × 166	(44', 17, 16', 29) (11, 21) (15, 2) (13) (39', 10) (25') (19')

Ratio Dimensions	Code
1 : 2 100 × 200	(52', 34, 31') (23', 16) (48', 8) (14', 9') (17') (32', 6) (20')
1 : 2 104 × 208	(39, 17, 58', 36) (9, 8) (2', 4) (13) (46', 14') (32') (26')
1 : 2 124 × 248	(66', 24, 46') (1, 15, 38') (25) (58', 16) (11, 4) (40') (26')
5 : 11 1005 × 2211	(520', 360', 275) (160', 200') (485', 195') (155', 45') (170') (290', 60') (230')
1 : 3 108 × 324	(41', 56', 33) (27', 14') (13', 1') (52', 15) (40') (2', 9') (7')
1 : 3 111 × 333	(59', 27', 13', 12') (1', 33) (14') (41') (52', 7') (15, 2') (43')
1 : 3 120 × 360	(67', 27, 44') (21', 23') (16', 9, 2') (53', 14', 25') (39') (28')
1 : 3 224 × 672	(87', 124', 39) (66', 21') (15, 16') (7', 18) (100', 47') (71') (53')
1 : 3 320 × 960	(172', 89', 59') (30', 87) (119') (144', 39, 33) (21, 112') (18') (31')
3 : 11 507 × 1859	(279', 129', 57', 42') (15', 99) (72') (201') (228', 51') (177', 75') (102')
1 : 7 248 × 1736	(137', 104', 21, 28) (59', 45') (111', 26') (2', 7) (47') (85') (52')
1 : 7 320 × 2240	(191', 8', 117', 28) (35, 21) (38', 79') (41') (129', 62') (124') (67')
1 : 7 400 × 2800	(236', 107', 57') (50', 49) (157') (164', 71', 7) (21', 136') (22') (93')

References

- [1] C. J. Bouwkamp, On the dissection of rectangles into squares ii, Kon. Nederl. Akad. Wetensch. 50 (1947) 58–71; (= Indag. Math. 9 (1947) 43–56).
- [2] R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares, Duke Math. J. 7 (1940) 312–340.
- [3] C. H. Jepsen, Dissections into 1 : 2 rectangles, Discrete Math. 148 (1996) 107–117.
- [4] C. H. Jepsen, Dissections of $p : q$ rectangles, Math. Comp. 65 (1996) 771–778.