Review Problems for Exam 3

The third exam will cover section 7.8, and chapter 9. Chapter 9 has been more theoretical than previous chapters. You should be able to explain the following:

1. For a linear system $\mathbf{x}' = A\mathbf{x}$, how does the behavior of the solutions depend on the eigenvalues. For example, if both of the eigenvalues are positive, we know the origin is a node. Why is this? Why do we get a spiral when we have complex eigenvalues? (This is really more from chapter 7, but it's important background for chapter 9).

2. Explain how we use linear systems to understand nonlinear systems. That is, how do we justify using the derivative matrix at a critical point in order to approximate the nonlinear system. How and why does this work? Go into detail, as if you were explaining this to someone who had taken calculus, but who hadn't seen this method before.

3. In the competing species equations, there are various ways to tell whether the coexistence critical point is a node or a saddle. Discuss these, and explain why they work. Also, explain mathematically why you never get a spiral point in the competing species equations.

4. Explain how Liapunov functions can tell us whether a critical point is stable, asymptotically stable, or unstable.

5. We discussed two methods for showing that a region does not contain a periodic solution to a system of differential equations. What are these, and why do they work? (You don't need to give rigorous mathematical arguments here, since we didn't see rigorous arguments in class. But, you should be able to convey the idea of what's going on.)

Here are some computational equations:

1. (7.8 #4) Find the general solution of the system. Also, sketch a few trajectories, and describe how solutions behave as $t \to \infty$.

$$\mathbf{x}' = \begin{pmatrix} -3 & \frac{5}{2} \\ -\frac{5}{2} & 2 \end{pmatrix} \mathbf{x}$$

2. (9.2 #20) Describe the trajectories of the following nonlinear system by finding an equation that they satisfy:

$$x' = 2x^2y - 3x^2 - 4y, y' = -2xy^2 + 6xy$$

3. (9.3 #10) Determine the critical points, find the corresponding linear systems, and classify the critical points.

$$dx/dt = x + x^2 + y^2, \quad dy/dt = y - xy$$

4. (9.4 #5) Find critical points, classify the corresponding linear systems, sketch approximate trajectories, determine the limiting behavior of x and y as $t \to \infty$, and interpret the results in terms of populations for the competing species model:

$$x' = x(1 - x - y), y' = y(1.5 - y - x)$$

5. (9.5 #3) Do the same as above for the following equations in the predator-prey model:

$$x' = x(1 - .5x - .5y), y' = y(-.25 + .5x)$$

6. (9.6 #8) Consider the equation

$$u'' + c(u)u' + g(u) = 0$$

where $g(u)$ is positive for small positive values of u, negative for small negative values of u, and $g(0) = 0$. Also, suppose that $c(u) \geq 0$ for all u near 0. Show that the point $u = 0, u' = 0$ is a stable critical point. (If you need a hint, see me!).

7. (9.7 Example 1) Show that the soystem below has a limit cycle by converting the system into polar coordinates. (Find r' and θ').

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x + y - x(x^2 + y^2) \\ -x + y - y(x^2 + y^2) \end{pmatrix}.$$