Real Analysis Background

You should know and understand the definitions, theorems, and proofs given below. If you have not seen this material, please take the time to learn what is presented below in the first week of class.

(1) **Limits and Continuity**

(a) If $\{x_n\}$ is a sequence of numbers, then we say $\lim_{n\to\infty} x_n = x$ if for any $\varepsilon > 0$, there exists a number N so that if $n \geq N$, then $|x_n - x| < \varepsilon$. We say that the sequence $\{x_n\}$ converges (or is convergent).

(b) If f is a function, we say $\lim_{x\to x_0} f(x) = L$ if for any $\varepsilon > 0$, there exists a δ so that if $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. If a function is only defined on a subset of $\mathbb{R}$, then the given condition only needs to hold for x in the domain of f.

(c) Equivalently, $\lim_{x\to x_0} f(x) = L$ if for any sequence x_n such that $\lim_{n\to\infty} x_n = x$, we have $\lim_{n\to\infty} f(x_n) = L$.

(d) A function is continuous at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.

(e) A function is continuous on an interval $[a, b]$ if it is continuous at every point on that interval.

(f) Theorem: If f and g are continuous on an interval $[a, b]$, then $f+g$, $f-g$, fg are also continuous on this interval. Also, f/g is continuous at all points except where g vanishes. Also, the composition of two continuous functions is continuous, wherever it is defined. These can all be proved from the $\varepsilon - \delta$ definition of the limit. You should know how to prove that $f+g$ and $f-g$ are continuous (provided f and g are), but I wouldn't expect you to remember the arguments for products, quotients, or compositions.

(g) A sequence $\{x_n\}$ is Cauchy if for any $\varepsilon > 0$, there is a number N so that for any pair of numbers n, m both greater than N, $|x_n - x_m| < \varepsilon$.

(h) Theorem: any Cauchy sequence of real numbers converges, and any convergent sequence of real numbers is Cauchy. This is probably the most important property of real numbers. It can be proved using the Least Upper Bound property of real numbers, but knowing the proof of the statement isn't so important for this class as knowing the statement.

(i) A sequence of functions f_n converges to a function f on an interval if for each x in the interval, the sequence of numbers $f_n(x)$ converges to $f(x)$.

(j) A sequence of functions f_n converges uniformly to a function f on an interval if for any $\varepsilon > 0$, there exists a number N so that whenever $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$, regardless of x. This, of course, implies that f_n converges to f, but not conversely.

(k) Theorem: If a sequence of continuous functions f_n converges uniformly to a function f, then f must be continuous. That is, the uniform limit of continuous functions is continuous. This is proved using what is known as an $\varepsilon/3$ argument. You should know and understand the proof: Suppose a is in the domain, and suppose given $\varepsilon > 0$. Choose N so that for $n \geq N$, $|f_n(x) - f(x)| < \varepsilon/3$ for all x in the domain (this can be done by uniform convergence). Now choose δ so that $|f_N(x) - f_N(a)| < \varepsilon/3$ whenever $|x - a| < \delta$, which can be done since f_N is continuous. Now, if $|x - a| < \delta$, then

$$\begin{aligned} |f(x) - f(a)| &= |f(x) - f_N(x) + f_N(x) - f_N(a) + f_N(a) - f(a)| \\ &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| \end{aligned}$$

by the triangle inequality. This is bounded by $\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

(2) **Series**

(a) Given a sequence of real numbers x_n, the sum $\sum_{n=1}^{\infty} x_n$ is equal to the limit of the partial sums $\sum_{n=1}^{N} x_n$ as N goes to infinity, if that sum exists. If it does, then the series $\sum_{n=1}^{\infty} x_n$ is said to converge. Otherwise, it is said to diverge. We often abbreviate $\sum_{n=1}^{\infty} x_n$ as $\sum x_n$. If the sequence x_n begins at something like x_2 or x_3, then $\sum x_n$ must be interpreted accordingly.

(b) A series $\sum_{n=1}^{\infty} x_n$ converges if and only if for any ε, there is a number N so that if $k > j > N$, then $|\sum_{n=j+1}^{k} x_n| < \varepsilon$. This is called the Cauchy convergence criterion, and follows immediately from the definition of series convergence coupled with the fact that convergent sequences are the same as Cauchy sequences.

(c) The series $\sum_{n=m}^{\infty} x_n$ is said to converge absolutely if the corresponding series $\sum_{n=m}^{\infty} |x_n|$ converges. Absolute convergence of a series implies convergence of that series. This can be proved using the Cauchy convergence criterion together with the triangle inequality.

(d) If the series $\sum_{n=1}^{\infty} x_n$ converges, then the sequence x_n must converge to 0. This is easy to prove using the Cauchy convergence criterion. It gives a necessary (but not sufficient) criterion for convergence. Thus, if x_n does not converge to 0, we can conclude immediately that the series $\sum x_n$ diverges.

(e) Comparison test: If $\sum v_k$ converges and $0 \le u_k \le v_k$ for each k, then $\sum u_k$ converges and $0 \le \sum u_k \le \sum v_k$. This again can be proved using the Cauchy convergence criterion.

(f) A power series is a series of the form $\sum a_k(x - x_0)^k$, where x is a variable.

(g) If a power series $\sum a_k(x - x_0)^k$ converges when $x = x_1$, then it converges absolutely for all x such that $|x - x_0| < |x_1 - x_0|$. (This follows from the comparison test and the geometric series; we'll review the proof when we get to the complex analogue).

(h) If a power series $\sum a_k(x - x_0)^k$ converges whenever $|x - x_0| < R$ but diverges for $|x - x_0| > R$, then R is called the radius of convergence of the power series. The series will converge absolutely for $|x - x_0| < R$, by the previous remark.

(i) Weierstrass M-test. Suppose given a sequence f_n of functions on an interval and numbers M_n so that $|f_n(x)| < M_n$ for all x in the interval and all numbers n. If the series $\sum M_n$ converges, then the power series $\sum f_n(x)$ converges absolutely and uniformly on the interval.

(3) **Differentiable functions**

(a) A function f is called differentiable on an interval (a, b) if the derivative $f'(x)$ exists and is continuous on (a, b). It follows that differentiable functions are continuous.

(b) If f is differentiable on (a, b), then for any $x_0 \in (a, b)$, there is a function $E(x)$ such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + E(x)$$

Moreover, $\lim_{x \to x_0} \frac{E(x)}{x - x_0} = 0$.

(c) Inverse Function Theorem: If $f(x)$ is a continuous function on an interval $[a, b]$ which is differentiable on (a, b), and its derivative is positive on (a, b), then there is a continuous function g defined on $[f(a), f(b)]$ which is an inverse to f in the sense that $f(g(x)) = x$ for all $x \in [f(a), f(b)]$ and $g(f(x)) = x$ for all $x \in [a, b]$. Moreover, the g is differentiable on $(f(a), f(b))$, and $g'(x) = \frac{1}{f'(g(x))} > 0$ for each $x \in (f(a), f(b))$.

(4) **Integrals**

(a) If f is a real-valued function on an interval $[a, b]$, and $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, and for each i with $1 \le i \le n$, x_i^* is a point between x_{i-1} and x_i, then the sum

$$\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$

is called a Riemann sum for f. The sequence $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ is called a partition of the interval $[a, b]$, and the maximum of $x_i - x_{i-1}$ over $1 \le i \le n$ is called the mesh of the partition. Thus, the above Riemann sum could be called the Riemann sum associated to the partition $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ with sample points $x_1^*, x_2^*, \cdots, x_n^*$.

(b) Suppose there is a number L so that for any $\varepsilon > 0$, there exists a number δ so that any Riemann sum associated to a partition whose mesh is less than δ is within ε of L, then L is called the Riemann integral of f over the interval $[a, b]$, and denoted $\int_a^b f(x)\, dx$.

(c) If f is a continuous function on an interval $[a, b]$, and $|f(x)| \le M$ for all $x \in [a, b]$ then

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx \le M(b - a).$$