9/5

Consider a system of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Suppose $b_1 = b_2 = \cdots = b_m = 0$. Suppose $(s_1, s_2, \ldots, s_n)$ and $(t_1, t_2, \ldots, t_n)$ are solutions. Show that

$$(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$$

is also a solution.

9/10

Suppose that $\vec{a}$ and $\vec{b}$ are in Span $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ and $\lambda$ and $\mu$ are real numbers. Show that $\lambda\vec{a} + \mu\vec{b}$ is in Span $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. (Use the algebraic properties of vectors.)

9/14

Suppose that $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are all solutions of $A\vec{x} = \vec{0}$. Suppose $\vec{x}$ is in Span $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$. Show that $A\vec{x} = \vec{0}$.

9/17

Suppose $A$ is an $n \times n$ matrix whose column vectors are linearly independent. Prove that for any vector $\vec{b}$ in $\mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a solution.

9/19

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, and the only vector $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{0}$ is the zero vector. Suppose $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p$ are linearly independent in $\mathbb{R}^n$. Prove that $T(\vec{v}_1), T(\vec{v}_2), \cdots, T(\vec{v}_p)$ are linearly independent in $\mathbb{R}^m$.

9/26

Suppose $A$ is an $m \times n$ matrix and $B$ is an $n \times p$ matrix. Prove that $(AB)^T = B^T A^T$.

10/1

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation which is both one-to-one and onto. Without referring to matrices, show that there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(S(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Show that $S$ is a linear transformation. Finally, show that $S(T(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

10/3

Suppose $A, B, C$, and $D$ are all $n \times n$ matrices, and each pair of these matrices commutes. (That is, $AB = BA$, $AC = CA$, $AD = DA$, etc.). Suppose $AD - BC$ is an invertible $n \times n$ matrix. Prove that $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is an invertible $2n \times 2n$ matrix.

10/8

As in Example 3, let $\mathbb{S}$ be the space of doubly infinite sequences of numbers. Let $H$ be the subset of $\mathbb{S}$ consisting of sequences

$$\{\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots\}$$

such that $y_{n+1} = y_n + y_{n-1}$ for every integer $n$. Show that $H$ is a subspace of $\mathbb{S}$.

10/10

Suppose $T : V \to W$ is a linear transformation, and $U$ is a subspace of $W$. Let $T^{-1}(U)$ be the set

$$\{\vec{v} \in V : T(\vec{v}) \in U\}.$$

Prove that $T^{-1}(U)$ is a subspace of $V$.
Next, if $U$ is a subspace of $V$, let $T(U)$ be the set

$$\{\vec{x} \in W : \vec{x} = T(\vec{u}) \text{ for some } \vec{u} \in U\}.$$

Show that $T(U)$ is a subspace of $W$.

10/15

Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\}$ is a basis for a vector space $V$. Suppose $T : V \to V$ is a linear transformation. Show that there is a matrix $A$ so that

$$[T(\vec{x})]_{\mathcal{B}} = A[\vec{x}]_{\mathcal{B}}$$

for all $\vec{x} \in V$. (Hint: look back at Section 1.9).

10/29

Let $V$ be the vector space whose elements are sequences of real numbers $\{x_1, x_2, \ldots\}$ such that $x_{n+3} = 2x_{n+2} - x_{n+1} + 4x_n$ for all $n \geq 1$. Prove that the dimension of $V$ is 3. (Hint: consider three sequences, one beginning $1, 0, 0$, another beginning $0, 1, 0$, and a third beginning $0, 0, 1$.)

10/31

Do Problem 34 in Section 4.6 of your text.

11/5

Suppose $T : V \to W$ is a linear transformation between two vector spaces. Let $\vec{w} \in W$ and suppose $T(\vec{v}_p) = \vec{w}$. Show that the solutions of $T(\vec{x}) = \vec{w}$ can all be written in the form $\vec{v}_p + \vec{u}$, where $\vec{u} \in \text{Ker } T$. Show also that any vector of this form is a solution of $T(\vec{x}) = \vec{w}$.

11/9

Let $A$ be an $n \times n$ matrix, and let $C_{ij}$ be the $(i,j)$-cofactor of $A$. Let $B$ be the matrix whose $(i,j)$ entry is $C_{ji}$. Show that the diagonal entries of the product matrix $AB$ are all equal to the determinant of $A$. (If this takes you more than three lines, you're doing something wrong!)

11/12

Let $A$ be an $n \times n$ matrix, and let $C_{ij}$ be the $(i,j)$-cofactor of $A$. Let $B$ be the matrix whose $(i,j)$ entry is $C_{ji}$. Show that

$$AB = \det(A) \cdot I_n.$$

(You've already shown that the diagonal entries of $AB$ are equal to $\det(A)$, so now you need to show that the entires that are not on the diagonal are all 0. Try showing that each of these entries can be expressed as the determinant of an $n \times n$ matrix which has two rows that are the same.)

11/19

(Problem 27) Show that $\lambda$ is an eigenvalue of $A$ if and only if $\lambda$ is an eigenvalue of $A^T$.

11/26

Suppose $A$ is a $2 \times 2$ matrix with only one eigenvalue $\lambda$ equal to 0. Show that $A^2 = 0$.

11/28

The trace of a square matrix $A$ is the sum of its diagonal entries, $\sum A_{ii}$. Prove that if $A$ and $B$ are $n \times n$ matrices, then the trace of $AB$ is equal to the trace of $BA$. Use this to show that if $A$ is similar to $B$, then the trace of $A$ is equal to the trace of $B$.

11/30

Show that if $\vec{v}$ is an eigenvector of a real matrix $A$ with complex eigenvalue $\lambda = a + bi$ where $b \neq 0$, then the real and imaginary parts of $\vec{v}$ are linearly independent.

12/5

Problem 29. Let $W = \text{Span}\,\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$. Show that if $\vec{x}$ is orthogonal to $\vec{v}_j$ for each $j$ between 1 and $p$, then $\vec{x}$ is orthogonal to every vector in $W$.

12/7

Prove the three parts of Theorem 7 (see Exercise 25).

12/12

Problem 22. Let $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p$ be an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $T(\vec{x}) = \text{proj}_W \vec{x}$. Show that $T$ is a linear transformation.

12/14

Prove Theorem 14 (see Exercises 19-21 for hints).