Lecture notes for Section 7.5

Terminology and Principles:

1. A solution of a recurrence relation is a sequence $a_0, a_1, a_2, a_3, \cdots$, and this solution has an associated generating function

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots .$$

2. Using the recurrence relation, you can often solve for the function $g(x)$. Then writing your solution as a power series gives a solution of the recurrence relation.

Example 1.

First, let's look at $a_n = a_{n-1} + n^2, a_0 = 1$. Note that the recurrence relation is only valid when $n \geq 1$, since a_{-1} isn't defined.

Now, we let

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

I want to use the recurrence relation to rewrite a_n in this expression, but I can't do this for the term a_0. So, I'll bring it to the other side.

$$g(x) - 1 = g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n.$$

Now, I can use the recurrence relation to rewrite a_n.

$$g(x) - 1 = \sum_{n=1}^{\infty} (a_{n-1} + n^2) x^n.$$

Now, I'll break the sum up into two pieces:

$$g(x) - 1 = \left(\sum_{n=1}^{\infty} a_{n-1} x^n \right) + \left(\sum_{n=1}^{\infty} n^2 x^n \right)$$

Now, I'll shift indices with the left term on the right side of the equation, by replacing the index n by $n+1$ everywhere I see it:

$$\sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n+1=1}^{\infty} a_{(n+1)-1}x^{n+1} = \sum_{n=0}^{\infty} a_n x^{n+1} = x\left(\sum_{n=0}^{\infty} a_n x^n\right).$$

This is just $xg(x)$.

I also need to study the term $\sum_{n=1}^{\infty} n^2 x^n$. Recall that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, so taking the derivative of both sides, we get

$$\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2},$$

and taking the derivative again, we get

$$\sum_{n=0}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Multiplying the first equation by x and the second equation by x^2, we get

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2},$$

$$\sum_{n=0}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

Adding these, we get

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

Now, we put this all together:

$$g(x) - 1 = xg(x) + \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

So,

$$g(x)(1-x) = 1 + \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

So,

$$g(x) = \frac{1}{(1-x)^4}\left((1-x)^3 + x(1-x) + 2x^2\right).$$

Thus, we've found the generating function for the recurrence relation. But we can simplify this:

$$\begin{aligned} g(x) &= \frac{1}{(1-x)^4}\left((1-3x+3x^2-x^3)+(x-x^2)+2x^2\right) \\ &= \frac{1}{(1-x)^4}\left(-x^3+4x^2-2x+1\right). \end{aligned}$$

Now,

$$\frac{1}{(1-x)^4} = 1 + \binom{4}{1}x + \binom{5}{2}x^2 + \binom{6}{3}x^3 + \cdots$$

so we get

$$g(x) = \left(1 + \binom{4}{1}x + \binom{5}{2}x^2 + \binom{6}{3}x^3 + \cdots\right)(-x^3+4x^2-2x+1)$$

This tells us that

$$a_r = \binom{r+3}{r} - 2\binom{r+2}{r-1} + 4\binom{r+1}{r-2} - \binom{r}{r-3}.$$

It's certainly a lot of work, but it gets the job done, and it's more powerful than any of the other techniques we've seen.

Example 2

$$a_n = a_{n-1} + a_{n-2} + 2^n, a_0 = a_1 = 1.$$

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$. In order to apply the recurrence relation, we must first remove the first two terms from the power series.

$$g(x) - 1 - x = \sum_{n=2}^{\infty} a_n x^n.$$

Now apply the recurrence relation

$$\begin{aligned} g(x) - 1 - x &= \sum_{n=2}^{\infty}(a_{n-1} + a_{n-2} + 2^n)x^n \\ &= \sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n + \sum_{n=2}^{\infty} 2^n x^n \end{aligned}$$

Now, we need to make it so that the subscripts on the a terms are the same as the exponents on the x terms:

$$x\left(\sum_{n=2}^{\infty} a_{n-1}x^{n-1}\right) + x^2\left(\sum_{n=2}^{\infty} a_{n-2}x^{n-2}\right) + \sum_{n=2}^{\infty}(2x)^n$$

Now, we shift indices in the first two terms above:

$$x\left(\sum_{n=1}^{\infty} a_n x^n\right) + x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) + \sum_{n=2}^{\infty}(2x)^n$$

Now, we rewrite the first two terms in terms of $g(x)$:

$$x(g(x)-1) + x^2(g(x)) + \sum_{n=2}^{\infty}(2x)^n$$

Now, we turn our attention to the last term. Note that

$$\sum_{n=0}^{\infty}(2x)^n = \frac{1}{1-2x}.$$

So,

$$\sum_{n=2}^{\infty}(2x)^n = \frac{1}{1-2x} - (1+2x).$$

So, putting this all together, we get

$$g(x) - 1 - x = x(g(x)-1) + x^2(g(x)) + \frac{1}{1-2x} - 1 - 2x.$$

So,

$$g(x)(1-x-x^2) = 1 + x - x - 1 - 2x + \frac{1}{1-2x}.$$

Solving for $g(x)$, we get

$$g(x) = \frac{4x^2 - 2x + 1}{(1-2x)(1-x-x^2)}.$$