Lecture notes for Section 5.1

Terminology and Principles:

1. The **Addition Principle** says that if A and B are two disjoint sets, then $|A \cup B| = |A| + |B|$. Here $|A|$ denotes the size of a set A.

2. Let $A \times B$ denote the set of ordered pairs (a, b) with $a \in A$ and $b \in B$. The **Multiplication Principle** says that $|A \times B| = |A| \times |B|$.

Essentially, the addition principle tells us that if we want to count the number of ways *either* one thing could happen *or* another thing could happen, then we should add the number of ways each individual thing could happen. For example, if I have five different muffins and three different doughnuts, and I want to choose one of them, then I must choose *either* one of five muffins *or* one of three doughnuts. There are $3 + 5 = 8$ ways to do this.

The multiplication principle tells us that if we want to count the number of ways that one thing could happen *and* another thing could happen, then we should multiply the number of ways that each individual thing could happen. For example, if I have five different muffins and three different doughnuts, and I want to choose one of each, then I want to choose *both* one of the five muffins *and* one of the three doughnuts. There are $3 \times 5 = 15$ ways to do this.

Examples

1. How many 5 digit numbers are there?

   The first digit could be anything between 1 and 9. Each of the next four digits could be any digit. By the multiplication principle, there are $9 * 10^4 = 90000$ such numbers.

2. How many are even.

   Now the final digit must be $0, 2, 4, 6$ or 8. Again by the multiplication principle, there are $9 * 10^3 * 2 = 45000$ such numbers.

3. How many have exactly one 3?

   If the 3 comes first, then there are 9 choices for each of the remaining digits, yielding $3 * 9^4$ numbers that begin with 3 but have no other

3's. Otherwise, there are 4 possible choices for where to put the 3, and then there are 8 ways to choose the first digit, and 9 ways to choose the other 3 digits. So, there are $4 * 8 * 9^3$ numbers that have exactly one 3, but not in the first digit. By the addition principle, there are $3 * 9^4 + 4 * 8 * 9^3$ 5-digit numbers having exactly one 3.

4. How many are palindromes?

   Now there are 9 choices for the first digit, and 10 for the next two digits. After this, the last two digits are determined. So, there are $9 * 10^2 = 900$ palindromes.

5. How many outcomes are possible if two distinct dice are rolled two successive times?

   There are 6^2 possibilities for rolling two distinct dice once. So, there are $(6^2)^2 = 1296$ possible ways to roll two distinct dice twice.

6. What's the probability that each die shows the same value on the second roll as on the first?

   There are 36 ways to roll the two distinct dice once. Then there is only one way to roll the two distinct dice a second time if each die is to show the same value on the second roll as on the first. So, the probability of this happening is $36/1296 = 1/36$.

7. What's the probability that the sum is the same on both rolls?

   The number of ways to get a sum of 2 is 1, so the number of ways to get a sum of 2 twice is $1 \cdot 1$. The number of ways to get a sum of 3 is 2, so the number of ways to get a sum of 3 twice is $2 \cdot 2$. The number of ways to get a sum of 4 is 3, so the number of ways to get a sum of 4 twice is $3 \cdot 3$. Continuing in this way, we find that the number of outcomes in which the sum on both rolls is the same is

   $$1 + 4 + 9 + 16 + 25 + 36 + 25 + 16 + 9 + 4 + 1 = 146.$$

   So, the total probability is $146/1296$.

8. What is the probability that the sum is greater on the second roll than on the first?

   There are 146 ways to get the same sum on both rolls, as we saw above. So, there are $1296 - 146 = 1150$ ways to get different rolls. In half of the cases when the sums are different, (i.e. in 575 cases), the sum will be larger on the second roll than the first. Thus, the probability is 575/1296.

9. How many 5's are there if we list the numbers from 1 to 100000?

   The number 5 occurs in the 1's digit $10,000$ times. It appears in each of the five digits the same number of times. So, it appears $50,000$ times.

10. If two numbers from 1 to 100 are chosen at random, what is the probability that the difference is 15?

    If the first number is between 1 and 15 or between 86 and 100, then there is only one possibility for the second number. If the first number is between 16 and 85, there are two possibilities. So, in all, there are $30 + 2 * 70 = 170$ possibilities. The number of ways to choose the two numbers is 10000, so the probability is $170/10000 = 17/1000$.