Lecture notes for Section 2.2

Terminology and Principles:

1. A **Hamilton path** is a path in a graph that visits every vertex of the graph exactly once.

2. A **Hamilton circuit** is a circuit in a graph that visits every vertex of the graph exactly once.

So Hamilton paths and Hamilton circuits are like Euler trails and Euler cycles, except that in the Euler case, each *edge* must be used exactly once, while in the Hamilton case, each *vertex* must be used exactly once.

Unfortunately, we don't have an easy way to check if a Hamilton circuit exists in a graph. As usual, it seems harder to show that no Hamilton circuit in a given graph exists – it seems like you'd have to list all possibilities, and go through thousands of cases even for relatively small graphs.

The basic way to show that no Hamilton circuit could exist in a graph is to suppose that a Hamilton circuit did exist, and try to come up with a contradiction (like we did when we were showing that certain graphs are nonplanar!) So, we need to list some things that would have to be true in order for a Hamilton circuit to exist.

1. Suppose v is a vertex in a graph with degree 2. Then you can be certain that any Hamilton circuit must go through the two edges touching v.

2. Suppose v is a vertex of degree greater than 2, but we know that 2 edges touching v must be used in any Hamilton circuit. Then we know that *none* of the other edges could possibly be used in the Hamilton circuit. (Once the circuit goes through v, it will never come back to v).

3. Suppose that a certain collection of edges must be used to build a Hamilton circuit, but this collection of edges forms a proper subcircuit (i.e., a circuit that doesn't use all the vertices of the graph.) Then no Hamilton circuit can exist.

Let's look at some examples.

Problem 4a. (This is on page 68). Since b has degree 2, we know that ab and be must be used in the Hamilton circuit. But also vertex d has degree 2, so the circuit must use edges ad and de. But this forms a proper subcircuit, which doesn't go through vertex c. So, there can't be a Hamilton circuit.

Problem 4b. Since vertices d, e, and f have degree 2, the six vertical edges have to be used in our Hamilton circuit. We know that one of the edges ab or ac must be used.

If we use ab, then we've used two edges at a, so we can delete edge ac by the second principle above. But then we have to use edge bc since now c has degree 2. On the other hand, we can't use bc, since we've already used two edges at b.

Similarly, if we use ac, then we can delete ab. This forces us to use edge bc, since b now has degree 2. But then, we've used three edges incident to c.

Problem 4c. Here, we have to use bc, ch, ad, dg, be, eh, af, and fg, since c, d, e and f have degree 2. But this gives us 2 proper subcircuits, so by the third principle above, there can be no Hamilton circuit.

Now look at the graph in Figure 2.8 on page 65. Does this graph have a Hamilton circuit? It seems like it would be quite a headache to show that it doesn't, even using the principles above. Here's another way. The graph in Figure 2.8 is bipartite – one set of vertices is $\{a, c, e, g, i, k, m, n, o\}$ while the other is $\{b, d, f, h, j, l, p, q\}$. In order for a Hamilton circuit in a bipartite graph to exist, the number of vertices on one side of the graph must be equal to the number of vertices on the other. Can you see why? This shows that the graph in Figure 2.8 cannot have a Hamilton circuit.