FINITE TRIGONOMETRIC PRODUCT AND SUM IDENTITIES

MARC CHAMBERLAND

Abstract. Several product and sum identities are established with special cases involving Fibonacci and Lucas numbers. These identities are derived from polynomial identities inspired by the Binet formulas for Fibonacci and Lucas numbers.

In some recent papers [1], [2], [4], [5], [6], one finds product identities such as

$$\prod_{s=1}^{(n-1)/2} \left[2\cos\left(\frac{\pi s}{n}\right)\right] = 1,\ n \text{ odd} \tag{1.1}$$

and

$$\prod_{s=1}^{(n-1)/2} \left[3+2\cos\left(\frac{2\pi s}{n}\right)\right] = F_n,\ n \text{ odd} \tag{1.2}$$

where F_n is the n^{th} Fibonacci number. The goal of this note is to unify these and other beautiful product or sum formulas as special instances of three polynomial identities. While the three identities are special instances of the general equation

$$x^n - y^n = \prod_{r=1}^{n}\left(x - e^{2ir\pi/n}y\right),$$

the inspiration for the specialized cases arises from the familiar Binet formulas for Fibonacci numbers F_n and Lucas numbers L_n. This approach contrasts with those of previously cited papers which use lesser-known representations of the Fibonacci numbers.

The first formula is

$$\prod_{s=1}^{\lfloor (n-1)/2\rfloor} \left[1+(y^2-1)\cos^2\left(\frac{\pi s}{n}\right)\right] = \frac{1}{y}\left[\left(\frac{1+y}{2}\right)^n - \left(\frac{1-y}{2}\right)^n\right] \tag{1.3}$$

for each natural number n. To prove that the polynomials on each side of the equation are equal, it suffices to show that they have the same degree, the same zeros, and evaluate to the same non-zero value at one point. Formula (1.3) holds since both sides have degree $n-1$ (resp. $n-2$) for n odd (resp. n even), share zeros at $\pm i\tan(\pi s/n)$ for $s = 1, \ldots, \lfloor (n-1)/2\rfloor$, and evaluate to one at $y = 1$. Evaluating this formula at several values gives various identities.

Sometimes these may be simplified if one uses a double-angle formula.

$$y=\sqrt{5}: \qquad \prod_{s=1}^{\lfloor (n-1)/2 \rfloor}\left[3+2\cos\left(\frac{2\pi s}{n}\right)\right] = F_n$$

$$y=i: \qquad \prod_{s=1}^{\lfloor (n-1)/2 \rfloor}\left[-2\cos\left(\frac{2\pi s}{n}\right)\right] = \begin{cases} 0, & n\equiv 0 \bmod 4 \\ (-1)^{(n-1)/4}, & n\equiv 1 \bmod 4 \\ (-1)^{(n-2)/4}, & n\equiv 2 \bmod 4 \\ (-1)^{(n-3)/4}, & n\equiv 3 \bmod 4 \end{cases}$$

$$\begin{array}{r}\text{coefficient of}\\ \text{dominant term}\end{array}: \qquad \prod_{s=1}^{\lfloor (n-1)/2 \rfloor}\left[2\cos\left(\frac{\pi s}{n}\right)\right] = \begin{cases} \sqrt{n/2}, & \text{n even,} \\ 1, & \text{n odd} \end{cases}$$

$$\begin{array}{r}\text{coefficient of}\\ \text{next dominant term}\end{array}: \qquad \sum_{s=1}^{\lfloor (n-1)/2 \rfloor}\tan^2\left(\frac{\pi s}{n}\right) = \begin{cases} (n/2-1)(n-1)/3, & \text{n even,} \\ n(n-1)/2, & \text{n odd} \end{cases}$$

$$y\to 0: \qquad \prod_{s=1}^{\lfloor (n-1)/2 \rfloor}\sin\left(\frac{\pi s}{n}\right) = \sqrt{\frac{n}{2^{n-1}}}$$

$$y=\sqrt{3}i: \qquad \prod_{s=1}^{\lfloor (n-1)/2 \rfloor}\left[-1-2\cos\left(\frac{2\pi s}{n}\right)\right] = \begin{cases} 0, & n\equiv 0,3 \bmod 6 \\ 1, & n\equiv 1,2 \bmod 6 \\ -1, & n\equiv 4,5 \bmod 6 \end{cases}$$

$$\text{coefficient of } y^2: \qquad \sum_{s=1}^{\lfloor (n-1)/2 \rfloor}\cot^2\left(\frac{\pi s}{n}\right) = \frac{(n-1)(n-2)}{6}$$

$$y=3: \qquad \prod_{s=1}^{\lfloor (n-1)/2 \rfloor}\left[5+4\cos\left(\frac{2\pi s}{n}\right)\right] = \frac{1}{3}\left[2^n-(-1)^n\right]$$

These identities may be found or derived from formulas in Hansen[3]. Specifically, the second formula corresponds to (91.2.3), the third to (91.2.2), the fourth to (21.1.2), the fifth to (91.1.4), the sixth to (91.2.9), and the seventh to (30.1.2). Note that the right side of the eighth formula is always an integer.

Differentiating (1.3) gives

$$\frac{1}{y}+\sum_{s=1}^{\lfloor (n-1)/2 \rfloor}\frac{2y\cos^2\left(\frac{\pi s}{n}\right)}{1+(y^2-1)\cos^2\left(\frac{\pi s}{n}\right)} = \frac{n}{2}\,\frac{\left(\frac{1+y}{2}\right)^{n-1}+\left(\frac{1-y}{2}\right)^{n-1}}{\left(\frac{1+y}{2}\right)^{n}-\left(\frac{1-y}{2}\right)^{n}} \tag{1.4}$$

Specific choices yield

$$y=\sqrt{5}: \qquad 1+10\sum_{s=1}^{\lfloor (n-1)/2 \rfloor}\frac{\cos^2\left(\frac{\pi s}{n}\right)}{3+2\cos\left(\frac{2\pi s}{n}\right)} = \frac{nL_{n-1}}{2F_n}$$

$$y=i: \qquad \sum_{s=1}^{\lfloor (n-1)/2 \rfloor}\sec\left(\frac{2\pi s}{n}\right) = \begin{cases} (n-1)/2, & n\equiv 1 \bmod 4 \\ 0, & n\equiv 2 \bmod 4 \\ (-n-1)/2, & n\equiv 3 \bmod 4 \end{cases}$$

$$y=\sqrt{3}i: \qquad 2\sum_{s=1}^{\lfloor (n-1)/2 \rfloor}\frac{\cos^2\left(\frac{\pi s}{n}\right)}{4\cos^2\left(\frac{\pi s}{n}\right)-1} = \begin{cases} (n-1)/3, & n\equiv 1 \bmod 3 \\ (n-2)/6, & n\equiv 2 \bmod 3 \end{cases}$$

The second formula relates to Hansen's (26.1.1) and (26.1.2).

The following polynomial equation involves the tan function and a sum of odd powers:

$$\prod_{s=1}^{q}\left[1+x^2\tan^2\left(\frac{2\pi s}{2q+1}\right)\right]=\frac{1}{2}\left[(1+x)^{2q+1}+(1-x)^{2q+1}\right]. \tag{1.5}$$

This identity is proven with the same approach as before: both sides have degree $2q$, share zeros at $x=\pm i\cot(2\pi s/(2q+1))$ for $s=1,\ldots,q$, and evaluate to one when $x=0$. Special choices include

$$\begin{aligned}
x=\sqrt{5}: &\quad \prod_{s=1}^{q}\left[\frac{1}{4}+\frac{5}{4}\tan^2\left(\frac{2\pi s}{2q+1}\right)\right]=L_{2q+1}\\
x=1: &\quad \prod_{s=1}^{q}\left[2\cos\left(\frac{2\pi s}{2q+1}\right)\right]=\begin{cases}1, & q\equiv 0,3 \bmod 4\\ -1, & q\equiv 1,2 \bmod 4\end{cases}\\
x=i: &\quad \prod_{s=1}^{q}\left[\frac{1}{2}-\frac{1}{2}\tan^2\left(\frac{2\pi s}{2q+1}\right)\right]=\begin{cases}1, & q\equiv 0,3 \bmod 4\\ -1, & q\equiv 1,2 \bmod 4\end{cases}\\
\begin{array}{r}\text{coefficient of}\\ \text{dominant term}\end{array}: &\quad \prod_{s=1}^{q}\tan\left(\frac{2\pi s}{2q+1}\right)=\begin{cases}\sqrt{2q+1}, & q\equiv 0,3 \bmod 4\\ -\sqrt{2q+1}, & q\equiv 1,2 \bmod 4\end{cases}\\
\begin{array}{r}\text{coefficient of}\\ \text{next dominant term}\end{array}: &\quad \sum_{s=1}^{q}\cot^2\left(\frac{2\pi s}{2q+1}\right)=\frac{q(2q+1)}{3}\\
x=\sqrt{3}i: &\quad \prod_{s=1}^{q}\left[\frac{1}{4}-\frac{3}{4}\tan^2\left(\frac{2\pi s}{2q+1}\right)\right]=\begin{cases}1, & q\equiv 0,2 \bmod 3\\ -2, & q\equiv 1 \bmod 3\end{cases}
\end{aligned}$$

The fourth formula relates to Hansen (91.3.3).

Differentiating (1.5) gives

$$\sum_{s=1}^{q}\frac{2x\tan^2\left(\frac{2\pi s}{2q+1}\right)}{1+x^2\tan^2\left(\frac{2\pi s}{2q+1}\right)}=(2q+1)\frac{(1+x)^{2q}-(1-x)^{2q}}{(1+x)^{2q+1}+(1-x)^{2q+1}} \tag{1.6}$$

This produces the special cases

$$\begin{aligned}
x=\sqrt{5}: &\quad \sum_{s=1}^{q} \frac{\tan^2\left(\frac{2\pi s}{2q+1}\right)}{1+5\tan^2\left(\frac{2\pi s}{2q+1}\right)} = \frac{(2q+1)F_{2q}}{4L_{2q+1}} \\
x=1: &\quad \sum_{s=1}^{q} \sin^2\left(\frac{2\pi s}{2q+1}\right) = \frac{2q+1}{4} \\
x=i: &\quad \sum_{s=1}^{q} \frac{\tan^2\left(\frac{2\pi s}{2q+1}\right)}{1-\tan^2\left(\frac{2\pi s}{2q+1}\right)} = \begin{cases} -\frac{2q+1}{2}, & q \text{ even} \\ -\frac{1}{2}, & q \text{ odd} \end{cases} \\
\text{coefficient of x}: &\quad \sum_{s=1}^{q} \tan^2\left(\frac{2\pi s}{2q+1}\right) = (2q+1)q \\
x=\sqrt{3}i: &\quad \sum_{s=1}^{q} \frac{\tan^2\left(\frac{2\pi s}{2q+1}\right)}{1-3\tan^2\left(\frac{2\pi s}{2q+1}\right)} = \begin{cases} 0, & q\equiv 0 \bmod 3 \\ -(2q+1)/8, & q\equiv 1 \bmod 3 \\ -(2q+1)/4, & q\equiv 2 \bmod 3 \end{cases}
\end{aligned}$$

The last polynomial equation is similar to the second, but with even powers:

$$\prod_{s=1}^{q}\left[1+x^2\tan^2\left(\frac{\pi(2s-1)}{4q}\right)\right] = \frac{1}{2}\left[(1+x)^{2q}+(1-x)^{2q}\right]. \tag{1.7}$$

Special choices include

$$\begin{aligned}
x=\sqrt{5}: &\quad \prod_{s=1}^{q}\left[\frac{1}{4}+\frac{5}{4}\tan^2\left(\frac{\pi(2s-1)}{4q}\right)\right] = \frac{L_{2q}}{2} \\
x=1: &\quad \prod_{s=1}^{q}\left[2\cos\left(\frac{\pi(2s-1)}{4q}\right)\right] = \sqrt{2} \\
x=i: &\quad \prod_{s=1}^{q}\left[\frac{1}{2}-\frac{1}{2}\tan^2\left(\frac{\pi(2s-1)}{4q}\right)\right] = \begin{cases} 0, & q \text{ odd}, \\ (-1)^{q/2}, & q \text{ even}, \end{cases} \\
\begin{array}{r}\text{coefficient of}\\ \text{dominant term}\end{array}: &\quad \prod_{s=1}^{q} \tan\left(\frac{\pi(2s-1)}{4q}\right) = 1 \\
\begin{array}{r}\text{coefficient of}\\ \text{next dominant term}\end{array}: &\quad \sum_{s=1}^{q} \cot^2\left(\frac{\pi(2s-1)}{4q}\right) = q(2q-1) \\
x=\sqrt{3}i: &\quad \prod_{s=1}^{q}\left[\frac{1}{4}-\frac{3}{4}\tan^2\left(\frac{\pi(2s-1)}{4q}\right)\right] = \begin{cases} -1/2, & q\equiv 1,2 \bmod 3 \\ 1, & q\equiv 0 \bmod 3 \end{cases}
\end{aligned}$$

The second formula relates to Hansen's (91.2.6), the fourth formula with (91.3.5), and the fifth formula with (30.1.5).

Differentiating (1.7) gives

$$\sum_{s=1}^{q} \frac{x \tan^2\left(\frac{\pi(2s-1)}{4q}\right)}{1 + x^2 \tan^2\left(\frac{\pi(2s-1)}{4q}\right)} = q\frac{(1+x)^{2q-1} - (1-x)^{2q-1}}{(1+x)^{2q} + (1-x)^{2q}} \tag{1.8}$$

This produces the special cases

$$x = \sqrt{5} : \quad \sum_{s=1}^{q} \frac{\tan^2\left(\frac{\pi(2s-1)}{4q}\right)}{1 + 5\tan^2\left(\frac{\pi(2s-1)}{4q}\right)} = \frac{qF_{2q-1}}{2L_{2q}}$$

$$x = 1 : \quad \sum_{s=1}^{q} \sin^2\left(\frac{\pi(2s-1)}{4q}\right) = \frac{q}{2}$$

$$x = i : \quad \sum_{s=1}^{q} \frac{\tan^2\left(\frac{\pi(2s-1)}{4q}\right)}{1 - \tan^2\left(\frac{\pi(2s-1)}{4q}\right)} = -\frac{q}{2}, q \text{ even}$$

$$\text{coefficient of x} : \quad \sum_{s=1}^{q} \tan^2\left(\frac{\pi(2s-1)}{4q}\right) = (2q-1)q$$

$$x = \sqrt{3}i : \quad \sum_{s=1}^{q} \frac{\tan^2\left(\frac{\pi(2s-1)}{4q}\right)}{1 - 3\tan^2\left(\frac{\pi(2s-1)}{4q}\right)} = \begin{cases} -q/4, & q \equiv 0 \bmod 3 \\ -q/2, & q \equiv 1 \bmod 3 \\ 0, & q \equiv 2 \bmod 3 \end{cases}$$

The third formula relates to Hansen's (21.1.4).

References

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Department of Mathematics and Statistics, Grinnell College, Grinnell, IA 50112
E-mail address: `chamberl@math.grinnell.edu`