ROCK-PAPER-SCISSORS MEETS BORROMEAN RINGS

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1. Introduction

Directed graphs with an odd number of vertices n, where each vertex has both $(n-1)/2$ incoming and outgoing edges, have a rich structure. We were lead to their study by both the Borromean rings and the game rock-paper-scissors. An interesting interplay between groups, graphs, topological links, and matrices reveals the structure of these objects, and for larger values of n, extensive computation produces some surprises. Perhaps most surprising is how few of the larger graphs have any symmetry and those with symmetry possess very little. In the final section, we dramatically sped up the computation by first computing a "profile" for each graph.

2. Three Weapons

Let's start with the two-player game rock-paper-scissors or RPS(3). The players simultaneously put their hands in one of three positions: rock (fist), paper (flat palm), or scissors (fist with the index and middle fingers sticking out). The winner of the game is decided as follows: paper covers rock, rock smashes scissors, and scissors cut paper.

Mathematically, this game is referred to as a *balanced tournament*: with an odd number n of weapons, each weapon beats $(n-1)/2$ weapons and loses to the same number. This mutual dominance/submission connects RPS(3) with a seemingly disparate object: Borromean rings.

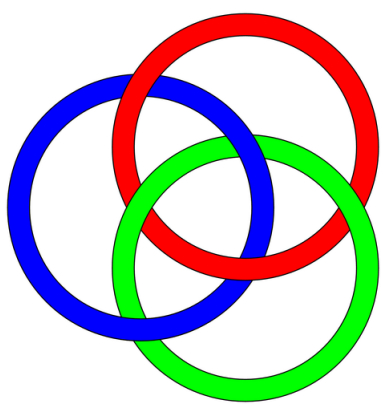
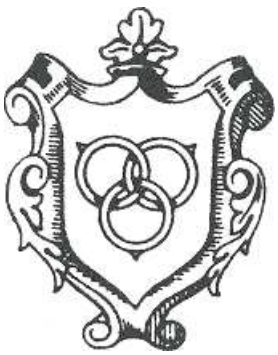


Figure 1. Borromean Rings.

The Borromean rings in Figure 1 consist of three unknots in which the red ring lies on top of the blue ring, the blue on top of the green, and the green on top of the red. Since each ring is above one ring and below another, balance is evident here as with RPS(3).

Versions of the Borromean rings have appeared in diverse cultures, from a symbol used in religion (Buddhist and Hindu temples and the Christian trinity) to company logos. The name comes from their use in the coat of arms of the aristocratic Borromeo family in Northern Italy; see Figure 2.



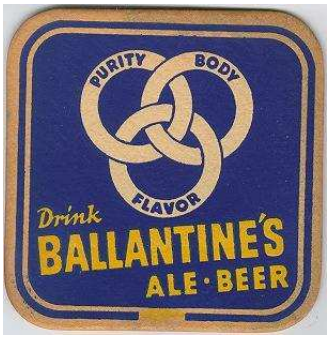


FIGURE 2. Borromean coat of arms, beer logo, and at a Shinto shrine.

How many different balanced tournaments are there? This is most easily seen — see Figure 3 — by representing the weapons as vertices in a directed graph, where flow in the edges indicate dominance. To ensure that balance is maintained, the only remaining possibility is to reverse the directions; so there are exactly two balanced tournaments. Another way to see this is to use the *automorphisms* of the graph, that is, the relabellings of the vertices that produce the same directed edges. Since, in Figure 3, these are the cycles (123), (132) and the identity permutation, the automorphism group of the graph is the cyclic subgroup C_3 of the permutation group S_3. This implies that the number of balanced graphs is $|S_3|/3 = 2$.

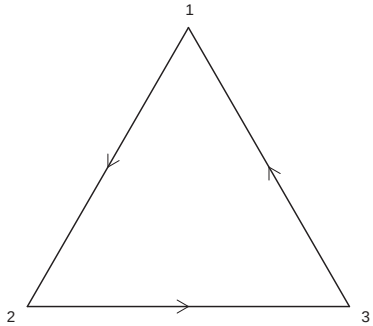


FIGURE 3. RPS(3) graph.

These two graphs, however, are essentially the same graph. One is obtained from the other by relabelling the vertices, that is, the graphs are *isomorphic*. This may be seen algebraically by using the adjacency matrices of the graphs. Construct the 3×3 matrix whose (i, j)-entry is 1 if an edge is directed from vertex i to vertex j and 0 otherwise. Thus, corresponding to each balanced tournament with n weapons there is a *balanced matrix*, that is, an $n \times n$ 0-1 matrix (with n odd) satisfying the following properties:

- each row and each column contains exactly $(n-1)/2$ ones with the rest of the entries zero,
- $a_{i,j} + a_{j,i} = 1$ whenever $i \neq j$.

The only 3×3 balanced matrices are A and A^T, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

The matrices correspond to the two graphs mentioned earlier. One is a relabelling of the other since the matrices are permutation similar:

$$A = PA^TP^{-1}$$

where P is the permutation matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Five Weapons

Rock-paper-scissors is much better known that its five-weapon cousin, rock-paper-scissors-lizard-Spock, which we denote as RPS(5). In this game, two new weapons are introduced: lizard (four fingers curled together and the thumb forming the mouth) and Spock (hand gesture used by Vulcans in *Star Trek*); see Figure 4 [3].

The five-weapon game was popularized by its mention on three episodes of the television show *The Big Bang Theory*. The game is clearly balanced. In the Borromean ring context, one can make five rings with the analogous dominance properties. Figure 5 shows both a computer-generated image and a solid made with Grinnell College's 3D printer. This model was displayed at a juried exhibition for the Bridges 2013 conference in Enschede, the Netherlands.

The graph representation of RPS(5) — shown in Figure 6 — is given with the standard design for K_5. How many distinct balanced graphs are there? To answer this question, note that each edge of the black pentagon is part of exactly one three-cycle while each edge of the red pentagon is part of exactly two three-cycles. Hence, any automorphism of the graph must map outer edges to outer edges and must retain their order. This implies the group of

FIGURE 4. Rock-paper-scissors-lizard-Spock.

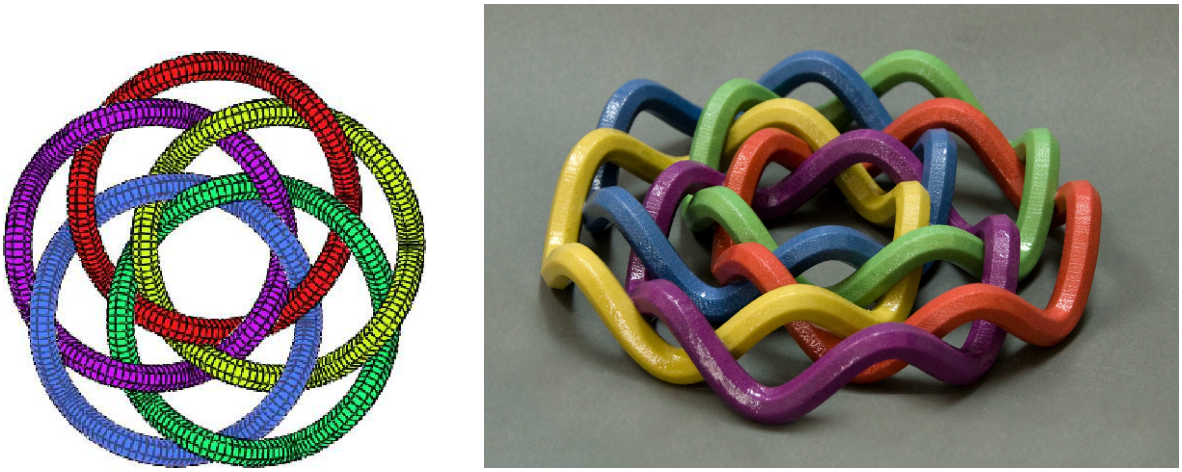


FIGURE 5. Borromean Five rings.

automorphisms is simply the cyclic subgroup C_5 of the permutation group S_5. The number of distinct RPS(5) graphs is therefore $5!/5 = 24$. Using the matrix perspective, one finds there are 24 5×5 balanced matrices. These matrices are all permutation similar, hence — up to relabelling — there is a unique RPS(5) graph.

An oft-noted property of the standard three Borromean rings is that removing any one ring frees the other two. This is an example of a Brunnian link, that is, a link where removing any one component frees all the other components. In the five-ring case, removing any one ring does not unlink the others. However, sometimes removing two rings does the trick. A careful analysis reveals that if two random rings are removed, there is a 50% chance

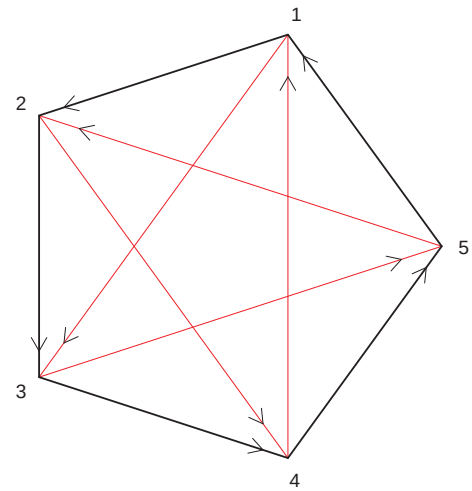


FIGURE 6. RPS(5) graph.

that the remaining three rings are freed. From this perspective, a new, fair two-player game could involve each player simultaneously removing a ring. If they have chosen distinct rings and they all fall apart, Player 1 wins, otherwise Player 2 wins.

4. Seven Weapons

Suppose now that one has seven weapons in a two-player game with each weapon beating three others and losing to the remaining three. We refer to this as RPS(7). Unlike RPS(3) and RPS(5), we will see that there are non-isomorphic RPS(7) games.

In computing the 7×7 balanced matrices, we also compute their characteristic polynomials, since matrices that are permutation similar have the same characteristic polynomial, although not conversely. We find only three distinct characteristic polynomials, and a further analysis shows that all 7×7 balanced matrices with the same characteristic polynomials are permutation similar. So there are exactly three non-isomorphic RPS(7) games.

As in the RPS(5) case, we seek the automorphism groups of each of the three cases. To this end, we produce representative graphs whose symmetries make their automorphism groups nearly transparent; see Figure 7. We refer to the three graphs as the ThreeSeptagons, the HexagonalPyramid, and the FanoPlane. The HexagonalPyramid gains its name by imagining the seventh vertex as positioned above the center of the hexagon.

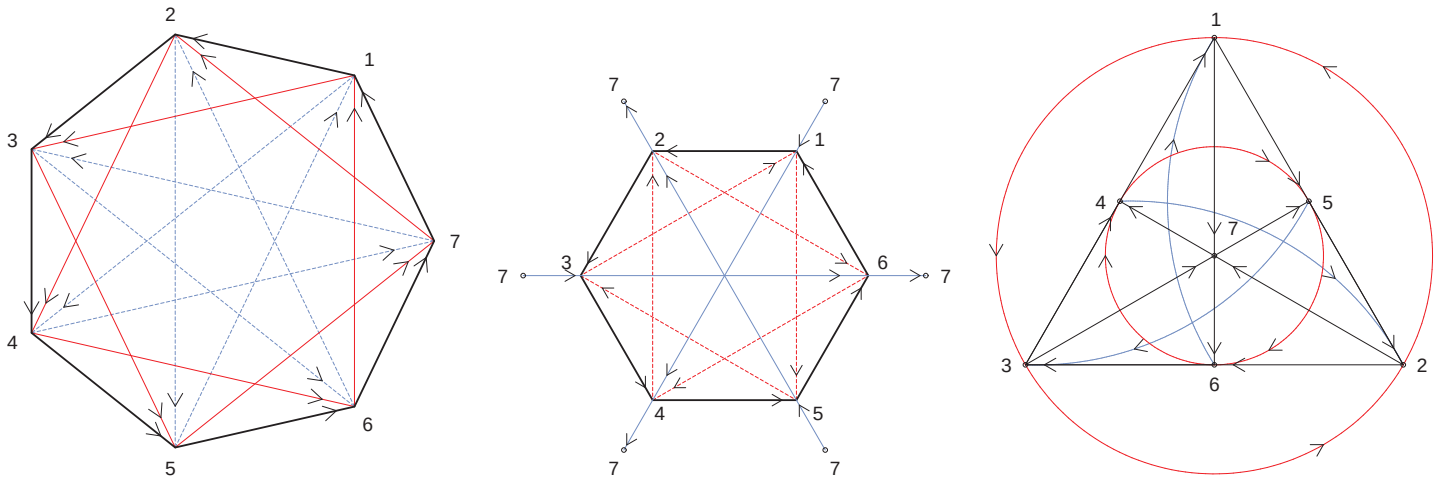


FIGURE 7. Three RPS(7) graphs.

To convince the reader that the three graphs are non-isomorphic, the corresponding matrices are

$$\begin{bmatrix} 0&1&1&1&0&0&0\\ 0&0&1&1&1&0&0\\ 0&0&0&1&1&1&0\\ 0&0&0&0&1&1&1\\ 1&0&0&0&0&1&1\\ 1&1&0&0&0&0&1\\ 1&1&1&0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&1&1&0&0\\ 0&0&1&0&0&1&1\\ 1&0&0&1&0&1&0\\ 0&1&0&0&1&0&1\\ 0&1&1&0&0&1&0\\ 1&0&0&1&0&0&1\\ 1&0&1&0&1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0&1&0&1\\ 1&0&0&0&0&1&1\\ 0&1&0&1&0&0&1\\ 1&1&0&0&1&0&0\\ 0&1&1&0&0&1&0\\ 1&0&1&1&0&0&0\\ 0&0&0&1&1&1&0 \end{bmatrix}.$$

The characteristic polynomials of these matrices are

$$(x-3)(x^2+x+2)^3,\ (x-3)(x^2+x+2)(x^4+2x^3+5x^2+4x+1),$$
$$(x-3)(x^6+3x^5+9x^4+13x^3+11x^2+5x+1)$$

To determine the number of distinct balanced graphs with $n = 7$, we calculate the automorphism groups of each of the three graphs. For the ThreeSeptagons graph, the analysis is similar to that for the graph in the $n = 5$ case. Each edge of the black septagon in that digraph is part of just one 3-cycle, each edge of the red septagon is part of two 3-cycles, and each edge of the blue septagon is part of three 3-cycles. Therefore, any isomorphism of the graph must map black edges to black edges in the same order. That is, the group of automorphisms is the cyclic subgroup C_7 of S_7, so the number of distinct digraphs isomorphic to the ThreeSeptagons is $|S_7|/7 = 720$.

For the HexagonalPyramid, the argument is a bit more delicate. Edges $(1, 2)$, $(3, 4)$, and $(5, 6)$ are part of three 3-cycles, the edges $(1, 4)$, $(3, 6)$ and $(5, 2)$ are part of just one 3-cycle, and all other edges are part of two 3-cycles. Vertex 7 has the unique property that every edge attached to it is part of

exactly two 3-cycles. Thus, any automorphism must map vertex 7 to itself and must map the edges $(1,2)$, $(3,4)$, and $(5,6)$ to themselves in the same order. That is, the group of automorphisms is the cyclic subgroup of S_7 generated by the product of 3-cycles $(135)(246)$. So the number of distinct RPS(7) digraphs isomorphic to the HexagonalPyramid is $|S_7|/|(135)(246)| = 7!/3 = 1680$.

For the FanoPlane graph, we use the fact that every edge is part of exactly two 3-cycles. Also, if (a,b) is an edge, then the 3-cycles it is part of can be written (a,b,c) and (a,b,d), where (c,d) is an edge; then (c,d) is part of (c,d,e) and (c,d,f), where (e,f) is an edge; and (e,f) is part of (e,f,a) and (e,f,b), which brings us back to edge (a,b). We use this pattern to construct all the automorphisms. Fix some edge, say $(3,4)$. Then any automorphism must map this edge to some other edge, say $(7,5)$. The pattern of edges (a,b), (c,d), (e,f) for these two edges is

$$(3,4),(1,5),(2,6) \quad \text{and} \quad (7,5),(3,2),(6,1),$$

respectively. Thus, an automorphism that maps 3 to 7 and 4 to 5 must map 1 to 3, 5 to 2, 2 to 6, and 6 to 1. The remaining vertex 7 must therefore map to the remaining vertex 4. That is, the unique automorphism mapping edge $(3,4)$ to $(7,5)$ is the 7-cycle (3745261). Similarly, for each of the twenty-one edges (a,b) of the FanoPlane graph, there is a unique automorphism that maps $(3,4)$ to (a,b), and these are all the automorphisms. So the number of distinct digraphs isomorphic to the FanoPlane is $|S_7|/21 = 240$. A further analysis would show that the automorphism group is the semidirect product of the cyclic group C_3 acting on C_7.

The Borromean rings corresponding to the three RPS(7) cases are displayed in Figure 8.

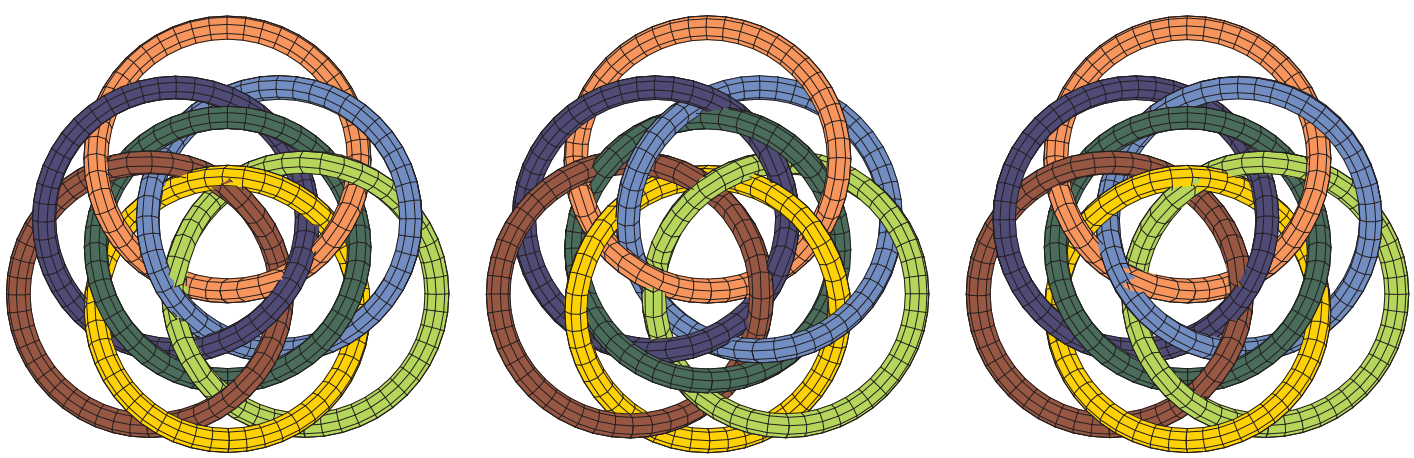


FIGURE 8. Three 7-ring configurations.

We saw that with five Borromean rings it is possible to remove two rings and have the rest fall apart. In the seven Borromean rings scenario, sometimes removing three rings allows the remaining four rings to disengage. However, the three non-isomorphic configurations behave differently on this count. For the ThreeSeptagons, the probability that the removal of three randomly chosen rings lets the remaining rings fall apart is 1/5. For the HexagonalPyramid, this reduces to 3/35, and for the FanoPlane, this reduces, remarkably, to zero. Here is a challenge: Can you find even one set of three rings in the middle figure that causes all the remaining rings to fall apart? Hint: Use the corresponding graph in Figure 7.

5. More and More Weapons

As n becomes larger, the number of non-isomorphic RPS(n) graphs increases very rapidly. We defer here to the computer scientists who have computed them up through $n = 13$; see [1, 2]. However, the 3-cycles that proved useful in the preceding sections also help us to compute efficiently the automorphism groups of all known RPS graphs.

Proposition 5.1. *Each edge of an RPS(n) graph is part of at least one 3-cycle.*

This property, whose proof we leave as an easy exercise for the reader, has useful interpretations in our other contexts. For any two rings in a ring configuration, there exists a third ring such that the set of three is a linked, Borromean ring configuration. Hence, to completely unlink any ring configuration, one must remove enough rings so that the corresponding graph has no 3-cycles. The interpretation for balanced matrices is that their third powers have no zero entries.

Theorem 5.2. *Every vertex of an RPS(n) graph is part of exactly* $(n^2-1)/8$ *3-cycles. The total number of 3-cycles in the graph is* $n(n^2-1)/24$.

Proof. Let v be any vertex in the graph, and let E denote the set of directed edges in the graph. Define

$$o(v) = \{w \mid (v, w) \in E\}, \qquad i(v) = \{x \mid (x, v) \in E\}$$

(the set of out-neighbors and in-neighbors, respectively, of v). The number of 3-cycles through v is just the number of edges directed from $o(v)$ to $i(v)$. Since $o(v)$ has $(n-1)/2$ vertices and each is the initial vertex of $(n-1)/2$ edges, the number of edges with initial vertex in $o(v)$ is $((n-1)/2)^2$. Furthermore, each pair of vertices in $o(v)$ is joined by a directed edge, and there are $\binom{(n-1)/2}{2}$ of them. Hence, the number of edges directed from $o(v)$ to $i(v)$ is $((n-1)/2)^2 - \binom{(n-1)/2}{2} = (n^2-1)/8$. The second claim follows by multiplying by $n/3$, which is the number of vertices divided by the number of times each 3-cycle is counted. □

We use these two results to help construct the "profile" of a graph. These profiles dramatically accelerate the computation of each graph's automorphism group. Given a vertex v in an RPS(n) graph G, we first define the profile p_v of that vertex. Given a 3-cycle $c =< v, w, x >$, let $p_{v,c}$ denote the set of triples $[\text{cycles}(v, w), \text{cycles}(w, x), \text{cycles}(x, v)]$, where $\text{cycles}(a, b)$ denotes the number of 3-cycles containing that edge. The proposition guarantees that the components of $p_{v,c}$ are never zero. The profile p_v is defined to be the multiset consisting of all the triples $p_{v,c}$. By the theorem, the multiset p_v contains $(n^2 - 1)/8$ elements. Now we can define $p(G)$, the profile of the graph G, by grouping together vertices v with the same profile p_v. Specifically, $p(G)$ is the set of ordered pairs $(\text{v-set}_k, p_k)$ where v-set_k is the set of vertices v such that $p_v = p_k$. In particular, the first components of these ordered pairs constitute a partition of the vertex set of G.

As an example, below is a typical profile for an RPS(9) graph. The prefix m in $m[a, b, c]$ indicates a triple $[a, b, c]$ that occurs m times in the multiset.

$$
\begin{array}{ll}
\{1\}: & \{1\,[1,4,4],\ 3\,[3,2,4],\ 6\,[3,3,2]\}, \\
\{2,3,4\}: & \{1\,[2,2,2],\ 1\,[2,2,3],\ 1,[2,3,3],\ 1\,[2,4,3],\ 3\,[3,2,3],\ 3\,[3,3,2]\}, \\
\{5,6,7\}: & \{1\,[2,2,2],\ 3\,[2,3,3],\ 1\,[3,2,2],\ 3\,[3,2,3],\ 1\,[3,3,2],\ 1\,[3,4,2]\}, \\
\{8\}: & \{6\,[2,3,3],\ 3\,[4,2,3],\ 1\,[4,4,1]\}, \\
\{9\}: & \{3\,[2,3,2],\ 3\,[2,3,4],\ 1\,[4,1,4],\ 3\,[4,3,2]\}.
\end{array}
$$

The first line claims that vertex 1 is the only vertex with the indicated profile, while the second line indicates that vertices 2,3, and 4 share the same profile. Note that since $n = 9$, the theorem asserts that each vertex is part of ten 3-cycles. This means that the 3-cycle total in the profile for each vertex equals 10. One sees this, for example, in the first line with 1+3+6 and in the second line with 1+1+1+1+3+3.

If G and H are both RPS(n) graphs, we say that $p(G)$ and $p(H)$ are *equal* if they have the same set of ordered pairs $(|\text{v-set}_k|, p_k)$. That is, we compare only the cardinalities of the vertex sets, not their contents. Thus, G and H are isomorphic if and only if $p(G) = p(H)$ and there is a permutation of the vertices of H that maps each vertex set of $p(H)$ to the corresponding vertex set of $p(G)$. In particular, a permutation of the vertices of G that is in the automorphism group of G maps each vertex set of $p(G)$ to itself.

To construct the automorphism group for the example above, one would normally need to check 9! permutations of the vertices. With the insight gained from the profile, however, one need only check $(3!)^2$ permutations, a significantly smaller number.

It would be reasonable to guess that as n increases, the quantity and complexity of the automorphism groups would grow dramatically. Remarkably, our data indicate that this is not the case. As n increases, the profiles of the graphs in RPS(n) tend to have finer and finer partitions of the vertex set. For $n = 13$, the majority of the profiles partition the vertex set into

singletons, and so the automorphism group of these graphs is immediately identified to be the trivial group $\{e\}$. In fact, for $n = 13$ almost all of the nearly one-and-a-half million automorphism groups were found in an average time of 0.012 seconds. There were only nine graphs whose groups were not computed so quickly. The few whose profile had just one vertex set — all 13 vertices had the same vertex profile, hence no permutations could be ruled out — required 64 hours of computing time per graph.

Below is a listing of the automorphism groups and their frequencies for the RPS(n) graphs, where $n = 9$, 11, and 13. The graph data were taken from [2]. The notation $N \rtimes H$ stands for the semidirect product of the subgroup H acting on the normal subgroup N.

# of $RPS(9)$ graphs	automorphism group
7	$\{e\}$
5	C_3
2	C_9
1	$(C_3 \times C_3 \times C_3) \rtimes C_3$
total: 15	

# of $RPS(11)$ graphs	automorphism group
1205	$\{e\}$
6	C_3
7	C_5
3	C_{11}
1	$C_3 \times C_3$
1	$C_{11} \rtimes C_5$
total: 1223	

# of $RPS(13)$ graphs	automorphism group
1494454	$\{e\}$
809	C_3
8	C_5
16	$C_3 \times C_3$
5	C_{13}
4	C_{15}
1	$C_{13} \rtimes C_3$
total: 1495297	

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