Dynamics of Maps with Nilpotent Jacobians

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Abstract. The *Discrete Markus-Yamabe Conjecture* (also known as the *LaSalle Conjecture*) imposed conditions on the Jacobian eigenvalues of a map in the hope of ensuring global attractivity of the fixed point. This paper pushes such assumptions to their extreme; the Jacobian is assumed to be nilpotent at all points. The dynamics of such maps is studied and diverse behavior is observed, from the quick collapse of points to a globally attractive fixed point, to maps with self-intersecting invariant curves.

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Running title: Nilpotent Jacobian Dynamics

1 Introduction

A now-settled problem known as the Discrete Markus-Yamabe Conjecture (or the LaSalle Conjecture) made claims about the dynamics of discrete maps $f : \mathbb{R}^n \to \mathbb{R}^n$ whose Jacobian matrix $Df(x)$ satisfies certain conditions. Specifically, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 map for which $f(0) = 0$ and the eigenvalues λ of $Df(x)$ satisfy $|\lambda| < 1$ for all x, must $x = 0$ be globally attractive? The condition on the eigenvalues of the Jacobian evaluated at the origin force the map to be attractive there, so it is not unreasonable to ask if global asymptotic stability ensues if the Jacobian eigenvalues maintain the same condition throughout the whole phase space.

Positive and negative results pertaining to this question can be found in [3].

The Markus-Yamabe conjecture is true when $n = 2$ and f is a polynomial. A two-dimensional rational counter-example is

$$f(x, y) = \left(\frac{-ky^3}{1 + x^2 + y^2}, \frac{kx^3}{1 + x^2 + y^2} \right), \ k \in \left(1, \frac{2}{\sqrt{3}}\right),$$

which admits a 4-cycle that includes the point $(\frac{1}{\sqrt{k-1}}, 0)$. A polynomial three-dimensional counter-example is

$$f(x, y, z) = \left(\frac{x}{2} + z(x + yz)^2, \frac{y}{2} + (x + yz)^2, \frac{z}{2} \right).$$

This map has a solution $(\frac{147}{32} \cdot 2^n, -\frac{63}{32} \cdot 2^{2n}, \frac{1}{2^n})$. The continuous counterpart to this conjecture, known as the Markus-Yamabe Conjecture, has a richer history; see, for example, [2], [4].

As with other Jacobian Conjectures, it is natural to impose extra conditions on the Jacobian matrix in the hope of attaining the desired end, in this case, global asymptotic stability. A stronger condition than requiring that the eigenvalues of the Jacobian have modulus less than one is that the Jacobian is nilpotent ie. the eigenvalues of the Jacobian are all zero. This paper examines this problem when $n = 2$ and $n = 3$. Section 2 considers the classification of nilpotent maps and disposes of some simple dynamics. Section 3 looks at the much richer dynamics of a class of three-dimensional nilpotent maps. Section 4 focuses on a particularly simple such map to show the complexity of the dynamics.

2 Classifying Nilpotent Maps

A classification of nilpotent maps has barely begun. The challenge of such a task is not surprising because of its relation to the Keller Jacobian Conjecture, a problem which has been reduced to considering *unipotent* maps. Unipotent maps are those whose Jacobian eigenvalues all equal one, so subtracting the identity map gives the nilpotent maps. We limit this study to the cases where $n = 2$ or $n = 3$.

The two-dimensional classification is settled.

Theorem 2.1 *(van den Essen [4])*
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be C^1. Then Df is nilpotent if $f = (u, v)$ has the form (up to an affine transformation):

$$\begin{aligned} u &= b\,\phi(ax + by) + c \\ v &= -a\,\phi(ax + by) + d \end{aligned}$$

for some constants $a, b, c, d \in \mathbb{R}$ and C^1 function $\phi : \mathbb{R} \to \mathbb{R}$.

The three-dimensional case is not complete. Two broad classes of functions have been found, both of which contain all known three-dimensional nilpotent maps.

Theorem 2.2 *(Chamberland and Essen [1])*
Let $f : \mathbb{R}^3 \to \mathbb{R}^3$. Then Df is unipotent if $f = (u, v, w)$ has one of the following forms (up to an affine transformation):

- *For some functions $a, b, c, d : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$*

$$\begin{aligned} u &= b(z)\,\phi(a(z)x + b(z)y, z) + c(z) \\ v &= -a(z)\,\phi(a(z)x + b(z)y, z) + d(z) \\ w &= 0 \end{aligned}$$

- *For some function $\phi : \mathbb{R} \to \mathbb{R}$*

$$\begin{aligned} u &= \phi(y - x^2) \\ v &= z + 2x\phi(y - x^2) \\ w &= -(\phi(y - x^2))^2 \end{aligned}$$

Since a nilpotent map f has $\det Df(x) = 0$ for all x, any region maps to a set with zero area. This allows trivial dynamics in some cases where all points are eventually fixed.

Theorem 2.3 *For the case $n = 2$, all points iterate to a unique fixed point in at most two iterations. For the first case when $n = 3$, one always reaches a fixed point in three iterates.*

Proof: In the case $n = 2$, the map is

$$\begin{aligned} u &= b\,\phi(ax + by) + c \\ v &= -a\,\phi(a(z)x + by + d \end{aligned}$$

Note that $au + bv = ac + bd$ is constant, so one reaches the fixed point $(b\,\phi(ac + bd) + c, -a\,\phi(ac + bd) + d)$ in at most two iterations. For the first case when $n = 3$, $z = 0$ after one iterate and one is reduced to the $n = 2$ case, so at most three iterates are required to reach the fixed point. $\square$

These maps have simple dynamics ultimately because the Jacobian has rank at most one. For some time, it was believed that all three-dimensional nilpotent maps had the form of the first class. However, van den Essen (see [1], [4]) found an example in the second class with rank 2. This was generalized to the second class noted above. The next section investigates their dynamics and finds a much more interesting story.

3 Dynamics with Rank 2 Jacobians

What remains is the second case for $n = 3$, namely, the iterative process

$$\begin{aligned} x_{k+1} &= \phi(y_k - x_k^2) && (1) \\ y_{k+1} &= z_k + 2x_k\phi(y_k - x_k^2) && (2) \\ z_{k+1} &= -(\phi(y_k - x_k^2))^2 && (3) \end{aligned}$$

Since $z_{k+1} = -x_{k+1}^2$, all points map instantly to the two-dimensional manifold $z = -x^2$. Restricting dynamics to that set reduces system (1)–(3) to the two-dimensional system

$$\begin{aligned} x_{k+1} &= \phi(y_k - x_k^2) && (4) \\ y_{k+1} &= -x_k^2 + 2x_k\phi(y_k - x_k^2) && (5) \end{aligned}$$

This system may be simplified yet further. Letting $a_k := \phi(y_k - x_k^2) - x_k$ gives

$$y_{k+1} = -(x_k - \phi(y_k - x_k^2))^2 + \phi(y_k - x_k^2)^2$$

$$= \quad -a_k^2 + x_{k+1}^2$$

thus

$$\begin{aligned}
a_{k+2} &= x_{k+3} - x_{k+2} \\
&= \phi(y_{k+2} - x_{k+2}^2) - \phi(y_{k+1} - x_{k+1}^2) \\
&= \phi(-a_{k+1}^2) - \phi(-a_k^2).
\end{aligned}$$

This produces the "reduced" system

$$a_{k+1} = b_k \tag{6}$$

$$b_{k+1} = \phi(-a_k^2) - \phi(-b_k^2) \tag{7}$$

The transformation is invertible since $x_{k+2} = \phi(-a_k^2)$ and $b_k = a_{k+1}$, so studying (6)–(7) is dynamically equivalent to studying (4)– (5). Note that system (6)–(7) does not change if a constant is added to ϕ, so **without loss of generality, let** $\phi(0) = 0$.

Theorem 3.1 *For the system (6)–(7),*

(a) the origin is the only fixed point, and it is super-attracting (ie. nearby points approach the origin in a quadratic way asymptotically).

(b) there are no 2-cycles

(c) if $\phi(-a^2) = -a$ for some constant $a \in \mathbb{R} \setminus \{0\}$, then $(0, -a)$ is part of a 3-cycle. Zero is always an eigenvalue of the Jacobian at $(0, -a)$.

(d) If ϕ is a polynomial, then the origin is not globally attractive.

(e) If

$$|\phi(-y^2) - \phi(-x^2)| < ||y| - |x|| \tag{8}$$

for all $x \neq y$, then the origin is globally attractive.

(f) The origin's basin of attraction is symmetric in all quadrants.

Proof: It is trivial to show that the fixed point is unique. The Jacobian matrix of the reduced system is nilpotent at the origin, hence the origin is supper-attracting. For part (b), suppose (x, y) was part of a 2-cycle. Then

$$\begin{aligned} x &= \phi(-y^2) - \phi(-x^2) \\ y &= \phi(-(\phi(-y^2) - \phi(-x^2))^2) - \phi(-y^2) \end{aligned}$$

This forces $y = \phi(-x^2) - \phi(-y^2) = -x$, hence $(x, y) = (0, 0)$, the fixed point, so there is no 2-cycle. For part (c), if $\phi(-a^2) = -a$, then

$$(0, -a) \to (-a, a) \to (a, 0) \to (0, -a),$$

so there is a 3-cycle. A quick calculation shows that zero is always an eigenvalue of the Jacobian evaluated at $(0, -a)$. If ϕ is a polynomial, then the equation $\phi(-x^2) = -x$ is an even equation. Since $\phi(0) = 0$, there must be another real solution, so part (c) shows that ther is not global asymptotic stabilty.

For part (e), consider the Liapunov function $V(x, y) = \max\{|x|, |y|\}$. Then condition (8) implies

$$\begin{aligned} V(u, v) &= \max\{|u|, |v|\} \\ &= \max\{|y|, |\phi(-y^2) - \phi(-x^2)|\} \\ &\leq \max\{|y|, ||y| - |x||\} \\ &\leq V(x, y) \end{aligned}$$

Another iterate forces a strict inequality, hence all points approach the origin.

The symmetry of origin's basin of attraction is clear since two iterates of (x, y), $(x, -y)$, $(-x, y)$ and $(-x, -y)$ all map to the same point. $\square$

Though the 3-cycle mentioned in part (c) cannot be repelling (because of the zero eigenvalue), it may be a saddle or an attractor. When $\phi(x) = cx$ for some non-zero constant c, the 3-cycle is always a saddle. If $\phi(x) = cx^2$, the 3-cycle may be either a saddle or an attractor.

4 The Simplest Map

Perhaps the simplest looking case where there is no global asymptotic stability is the case $\phi(x) = -x$. We already know there is a 3-cycle, but what else can be said about this map? It ends up that it exhibits some pretty wild behavior. The reduced map is explicitly given as

$$u = y \tag{9}$$

$$v = x^2 - y^2 \tag{10}$$

The Liapunov function used in the last section can be used here to show that the open square $(-1, 1) \times (-1, 1)$ is in the origin's basin of attraction.

This basin of attraction is also unbounded. The point $(1, 1)$ is on the basin boundary, so if it has a divergent predecessor set, by continuity the basin itself must be unbounded. Each point (u, v) has two predecessors, namely $(\sqrt{v + u^2}, u)$ and $(-\sqrt{v + u^2}, u)$. Letting $y_0 = 1$ and $y_{-1} = 1$ and using the relationship $y_{k+2} = y_k^2 - y_{k+1}^2$, positive predecessors evaluate to $y_{-2} = \sqrt{2}$, $y_{-3} = \sqrt{3}$, $y_{-4} \geq 2$. By induction, one may prove that $y_{-k} \geq \sqrt{k}$, hence the basin is unbounded.

A portion of the basin is shown in the figure 1. It is similar to the basin of attraction of the origin for the map

$$\begin{aligned} u &= x^2 + xy - y^2 \\ v &= \frac{1}{2}xy \end{aligned}$$

studied by Nien[7].

The stable manifold of the 3-cycle point $(0, 1)$ can be studied more closely. It is an even function $y = \psi(x)$ satisfying the functional equation

$$\psi(\psi(x)^2 - (x^2 - \psi(x)^2)^2) = (x^2 - \psi(x)^2)^2 - (\psi(x)^2 - (x^2 - \psi(x)^2)^2)^2$$

with $\psi(0) = 1$ and $\psi'(0) = 0$. The first few terms in a series expansion give

$$\psi(x) = 1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{5}{16}x^6 + \frac{51}{128}x^8 + \frac{137}{256}x^{10} + O(x^{12})$$

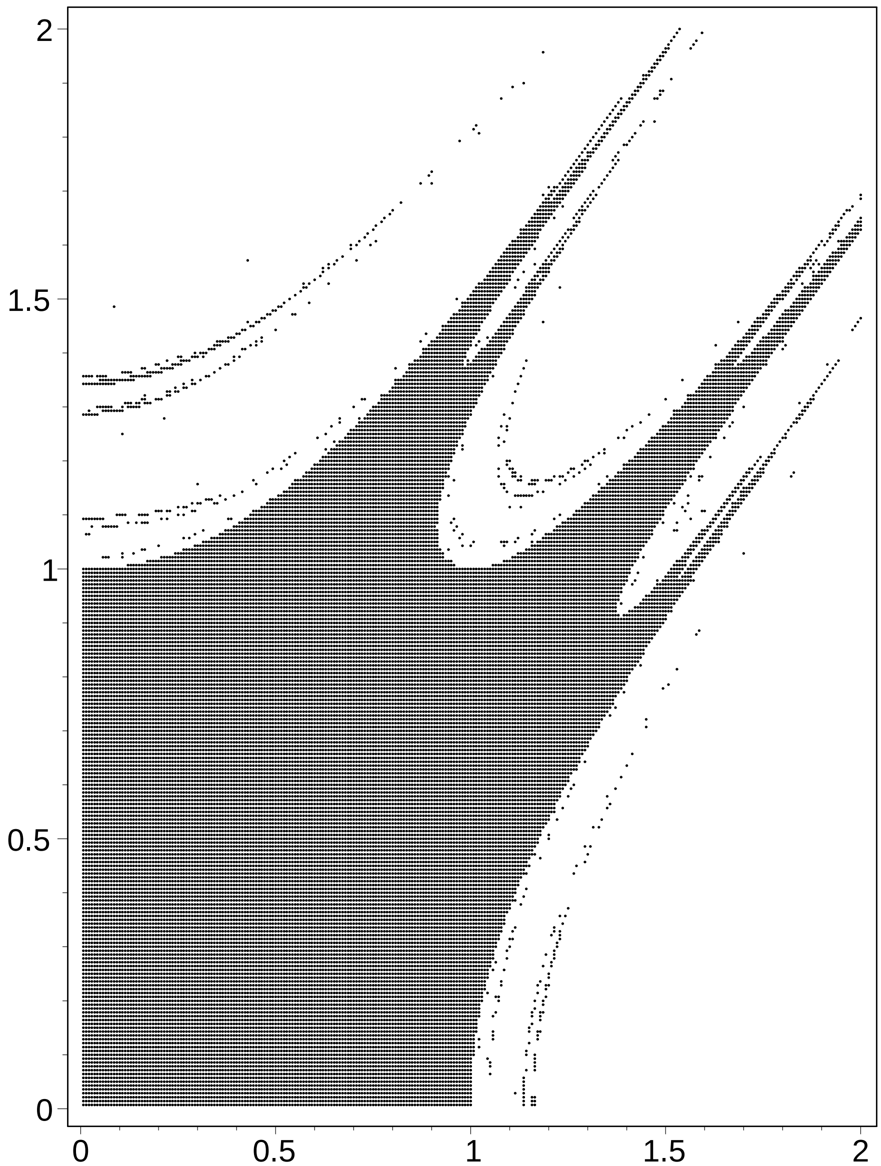

Figure 1: The basin of attaction in $[0, 2] \times [0, 2]$.

Since the eigenvalue corresponding to the stable manifold is zero, further calculation is required to determine the dynamics near $(0, 1)$. Three iterates of the map take $(x, \psi(x))$, with $|x|$ sufficiently small, to the right-hand plane.

It is easy to show that there are no other 3-cycles. To see this, note that a point which is part of 3-cycle must satisfy

$$
\begin{aligned}
x &= y^2 - (x^2 - y^2)^2 \\
y &= (x^2 - y^2)^2 - (y^2 - (x^2 - y^2)^2)^2
\end{aligned}
$$

One may manipulate these to show that $x, y \in \{0, 1\}$. Since there are also no other fixed points to the map, this implies all points on the stable manifold through $(0, 1)$ approach the 3-cycle. By considering the predecessor sets of this basin boundary curve, one obtains an infinite collection of curves, all of which form the boundary to the immediate basin of attraction of the origin.

The unstable manifold coming out of $(0, 1)$ goes through an amazing path and crosses itself; see figure 2. This implies that this invariant curve crosses itself infinitely often (there are no other fixed points to stop it). Moreover, since this invariant curve crosses the basin of attraction, there must be portions of the curve arbitrarily close to $(0, 1)$ which are also in the basin. This means the basin of attraction has infinitely many components.

Self-intersecting invariant manifolds have not been commonly noted in the literature. The earliest example can be found in [5, 201ff.], with the example

$$
\begin{aligned}
u &= y - \frac{1}{1000}y^3 + F(x) \\
v &= -x + F\left(y - \frac{1}{1000}y^3 + F(x)\right)
\end{aligned}
$$

where

$$F(x) = \frac{x}{2} + \frac{x^3}{1 + x^2}.$$

Such self-intersections are typical among non-invertible maps. More general properties may be found in [6].

5 Conclusion

Since counter-examples for the Discrete Markus-Yamabe have been found, one must impose stronger conditions on the Jacobian matrix in hopes of obtaining global stability. The paper studied the strongest possible imposition, the

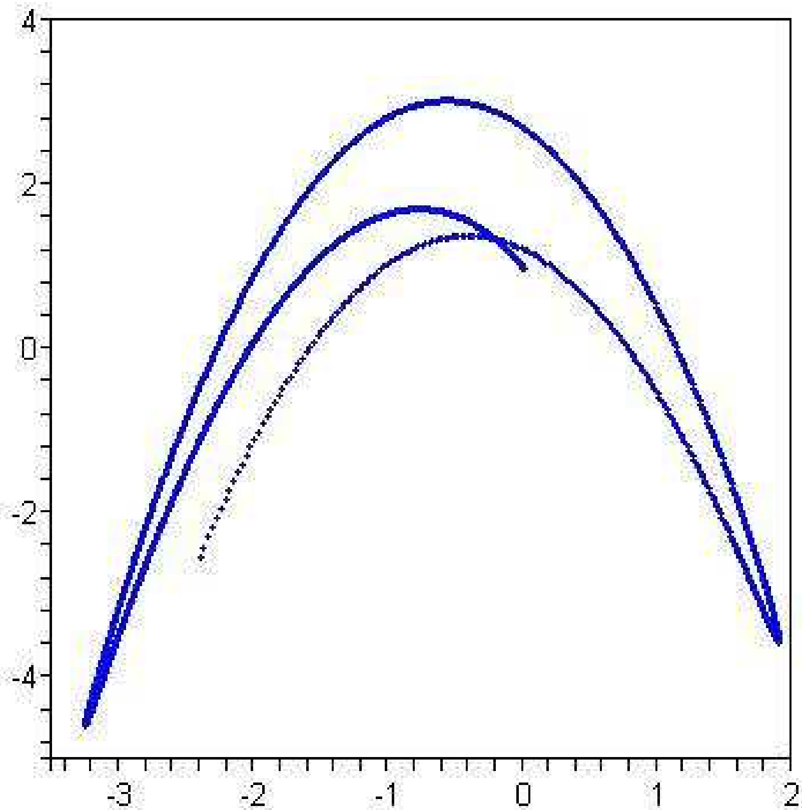

Figure 2: Unstable Manifold eminating from $(0, 1)$ with self-intersection.

nilpotency of the Jacobian. While the two-dimensional case and some three-dimensional cases yield a very strong global attractivity (attained in two or three iterations), another class of three-dimensional maps yields dynamics with cycles, divergent orbits, and self-intersecting invariant curves. Simply put, the strongest Jacobian conditions alone cannot yield global asymptotic stability.

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