Mean Value Integral Equations and the Helmholtz Equation [1]

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Abstract. This paper considers the mean-value functional equation

$$f(x) = \frac{1}{\mu(B_R(x))} \int_{B_R(x)} f(t)dt$$

where $B_R(x)$ is a ball of radius R centred at $x \in R^n$, and μ represents Lebesgue measure. By using the spectral synthesis, we find that the space of continuous functions satisfying this equation (in the case $n = 1, 2$) is characterized in terms of solutions of the Helmholtz equation.

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1 Introduction

The equation

$$f(x) = \frac{1}{\mu(B_R(x))} \int_{B_R(x)} f(t)dt \tag{1}$$

where μ represents Lebesgue measure, has received considerable attention (see the excellent survey on mean-value theorems by Netuka and Veselý[5]). Throughout the paper, R is a fixed positive constant, so the results obtained may be considered as "one-radius" theorems (one may see Zalcman[10] for a discussion on "two-radius" theorems). In the case $n = 1$, Walter[7] recently showed how to generate a large class of solutions to equation (1). Besides the linear functions, which are sometimes referred to as the trivial solutions, Walter remarked that exponential functions

$$f(x) = e^{\lambda x}$$

satisfy Equation (1) if and only if $\sinh(\lambda R) = \lambda R$. He also found that if $g(x)$ is a C^∞ function defined on $[-R, R]$ such that

$$g^{(j)}(-R) = g^{(j)}(0) = g^{(j)}(R) = 0, \quad j = 0, 1, 2, \ldots, \tag{2}$$

then g has a unique extension to the real line such that $f = g'$ satisfies Equation (1). As may be seen in Netuka and Veselý[5], many authors have found examples of non-harmonic functions which satisfy Equation (1) for all n.

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In this paper, I wish to show how to generate all the continuous solutions for Equation (1), at least in the cases $n = 1$ and $n = 2$. We shall use the method of spectral synthesis. In Section 2, the spectral synthesis method is explained and the results which will be used shall be introduced. Section 3 considers the case $n = 1$, while section 4 details the two-dimensional case.

2 Spectral Synthesis

Let $C(R^n)$ denote the space of continuous functions on R^n with the topology of uniform convergence of functions on compact sets, and $\mathcal{E}(R^n)$ the space of infinitely differentiable functions on R^n with the topology of uniform convergence of functions and their derivatives on compact sets. A function $f : R^n \to C$ is said to be a *polynomial-exponential function* if

$$f(x) = p(x)e^{iz\cdot x}$$

for all $x \in R^n$, where $p(\cdot)$ is a polynomial, and $z \in C^n$ is constant. In his study of mean-periodic functions of one variable, Schwartz[6] discovered the following result:

Theorem 2.1 *Every closed translation-invariant subspace of $C(R)$ or $\mathcal{E}(R)$ is spanned by the polynomial-exponential functions it contains.*

For some time, it was not clear whether this theory extended to higher dimensions. Gurevich[4] showed that translations were not sufficient to generate a spectral synthesis for dimensions greater than one. Brown, Schreiber and Taylor[2] showed that the extra bit needed to generalize the spectral synthesis to other Euclidean spaces was to include rotations, thus for the two-dimensional case, we have

Theorem 2.2 *([2]) Every closed translation-invariant rotation-invariant subspace of $C(R^2)$ or $\mathcal{E}(R^2)$ is spanned by the polynomial-exponential functions it contains.*

The following results will work in the space $\mathcal{E}(R^2)$, but Brown *et al.* noted that these results may extend to $C(R^2)$, due to the following density result (see [2]):

Theorem 2.3 *Let V be a closed translation-invariant subspace of $C(R^2)$, and let $V_1 = V \cap \mathcal{E}(R^2)$. Then V_1 is dense in V in the topology of $C(R^2)$.*

The following theorems are proven in Chamberland and Gladwell[3].

Theorem 2.4 *Let V be a closed translation-invariant rotation-invariant subspace of $\mathcal{E}(R^2)$. Suppose there exists a function $g(x) = e^{i\xi\cdot x}$ in V with $\xi = (\xi_1, \xi_2) \neq (0, 0)$, and let $\alpha = \xi_1^2 + \xi_2^2$. Then every function $u \in \mathcal{E}(R^2)$ satisfying the Helmholtz equation*

$$\Delta u + \alpha u = 0 \tag{3}$$

is in V.

We would like to complement Theorem 2.4 by showing when a space V is actually just the space of Helmholtz functions.

Theorem 2.5 *Let V be a closed translation-invariant rotation-invariant subspace of $\mathcal{E}(R^2)$. Suppose there exists a function $g(x) = e^{i\xi\cdot x}$ in V with $\xi = (\xi_1, \xi_2) \neq (0,0)$, and let $\alpha = \xi_1^2 + \xi_2^2$. Suppose that among all such functions g, the value of α is unique, and that the only polynomial-exponential functions in V with a linear polynomial part and exponential part $e^{i\xi\cdot x}$ are multiples of*

$$De^{i\xi\cdot x},$$

where $x = (x_1, y_1)$ and

$$D := y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1}.$$

Lastly, in the case $\alpha = 0$, we also impose that V contains no non-harmonic polynomials. Then V is precisely the space of solutions to

$$\Delta u + \alpha u = 0.$$

Finally, note that the preceding theorem may be naturally extended to consider spaces where α is not unique; if the conditions of Theorem 2.5 hold for each value of α, then the space V is the closed span of solutions of the Helmholtz equation (3) over all values of α.

3 The One-Dimensional Case

For the case $n = 1$, Equation (1) becomes

$$f(x) = \frac{1}{2R} \int_{x-R}^{x+R} f(t)dt. \tag{4}$$

Clearly the space of continuous functions satisfying Equation (4), which we denote by V, is closed and translation-invariant. We therefore may use Theorem 2.1 and claim that the polynomial-exponential functions which satisfy Equation (4) generate the space of functions. The pure exponential function $f(x) = e^{\lambda x}$ satisfies Equation (4) if and only if $\sinh(\lambda R) = \lambda R$. Walter remarked that by a theorem of Polya there is a countably infinite number of such λ. One easily sees that if $x^n e^{\lambda x}$ is in V (with $n \geq 1$), then $x^{n-1}e^{\lambda x}$ is also in V. If we suppose that $xe^{\lambda x}$ is a solution to Equation (4) with $\lambda \neq 0$, then

$$\begin{aligned} xe^{\lambda x} &= \frac{1}{2R}\int_{x-R}^{x+R} te^{\lambda t}dt \\ &= \frac{1}{2R\lambda}\left[(x+R)e^{\lambda(x+R)} - (x-R)e^{\lambda(x-R)} - \frac{1}{\lambda}\left(e^{\lambda(x+R)} - e^{\lambda(x-R)}\right)\right], \end{aligned}$$

thus

$$2R\lambda x = (x+R)e^{R\lambda} - (x-R)e^{-R\lambda} - \frac{1}{\lambda}\left(e^{R\lambda} + e^{-R\lambda}\right).$$

Using the fact that $\sinh(\lambda R) = \lambda R$, we obtain

$$2R\lambda x = 2R\lambda x + R\left(e^{R\lambda} + e^{-R\lambda}\right) - 2R,$$

thus

$$0 = \left(e^{R\lambda}/2 - e^{-R\lambda}/2\right)^2.$$

Having $e^{R\lambda}/2 = e^{-R\lambda}/2$ forces $\lambda = 0$, hence if $p(x)e^{\lambda x} \in V$ for some polynomial p and $\lambda \neq 0$, then p is a constant.

Similarly, we show that the polynomial x^2 cannot be in V:

$$\begin{aligned} x^2 &= \frac{1}{2R}\int_{x-R}^{x+R} t^2 dt \\ &= \frac{1}{6R}\left[6x^2R + 2R^3\right], \end{aligned}$$

which holds only if $R = 0$. This contradiction implies the only polynomials in V must be linear; it is clear that all the linear polynomials are in V.

In conclusion, we use Theorem 2.1 to obtain

Theorem 3.1 *The space of continous functions satisfying Equation (4) is the closed linear span of the linear functions and the exponential functions $e^{\lambda x}$, where* $\sinh(\lambda R) = \lambda R$.

It is worth noting that there is no result connecting the functions mentioned by Walter through Equation (2) with the span of exponential functions obtained. Walter showed that if $n(r)$ denotes the number of $|\lambda| < r$ such that $\sinh(\lambda R) = \lambda R$, then

$$n(r) = \frac{Rr}{\pi} + O(\log(r)),$$

which implies

$$\lim_{r\to\infty} \frac{n(r)}{r} = \frac{R}{\pi}. \tag{5}$$

Now we shall see how this relates to nonharmonic Fourier series. Let $\Lambda = \{\lambda_n\}$ be a sequence of real or complex numbers. The *completeness radius* of Λ is defined to be the number

$$S(\Lambda) = \sup\{A : \{e^{i\lambda_n t}\} \text{ is complete in } C[-A, A]\}.$$

Theorem 13 in Young[8, p.138] claims that if Λ is a sequence of positive *reals* satisfying condition (5), then $S(\Lambda) \geq R$. Unfortunately, the set $\{\lambda_n\}$ under consideration not only fail to be real, but have arbitrarily large imaginary parts. However, it is reasonable to speculate from Walter's condition (2) that for our system, we have $S(\Lambda) = R$. This suggests that Walter's result indirectly yields a new approach to determining the span of some families of exponentials.

4 The Two-Dimensional Case

We consider the equation

$$f(x) = \frac{1}{\pi R^2}\int_{B_R(x)} f(t)dt \tag{6}$$

where $x, t \in R^2$. As an aid to the integration we shall do, we introduce the Pizzetti formula[9]:

$$\int_{B_R(z)} u(t)dt = 2\pi R \sum_{k=0}^{\infty} \frac{\Delta^k u(z)}{k!(k+1)!} \left(\frac{R}{2}\right)^{2k+1}, \tag{7}$$

where $z \in R^2$. This is a generalization of the mean-value formula for harmonic functions. Again, let V denote the space of continuous functions satisfying Equation (6). The space V is clearly translation-invariant, rotation-invariant and closed in the usual topology. Now we seek for polynomial-exponential functions satisfying Equation (6).

First we note that any polynomials satisfying Equation (6) must be harmonic. Assume $f = f(x)$ is a nonlinear polnomial. Write f as

$$f(x) = \sum_{k=0}^{n} h_k(x)$$

where h_k is a homogeneous polynomial of degree k. Using the Pizzetti formula, and comparing the order of the terms, we see that $\Delta h_n = 0$. Using this inductively, we find that $\Delta h_k = 0$ for all k, hence f is harmonic.

Let us now consider non-trivial pure exponentials. Letting $f(x) = e^{i\lambda\cdot x}$, with $\lambda = (\lambda_1, \lambda_2) \in C^2$, we have

$$\Delta f + \omega^2 f = 0,$$

where $\omega = \sqrt{\lambda_1^2 + \lambda_2^2}$. If $\omega = 0$, then f is harmonic, so we assume that $\omega \neq 0$. Using Equation (7), we find that

$$\begin{aligned} \int_{B_R(x)} f(t)dt &= 2\pi R \sum_{k=0}^{\infty} \frac{\Delta^k f(x)}{k!(k+1)!} \left(\frac{R}{2}\right)^{2k+1} \\ &= 2\pi R \sum_{k=0}^{\infty} \frac{(-\omega^2)^k f(x)}{k!(k+1)!} \left(\frac{R}{2}\right)^{2k+1} \\ &= 2\pi R \frac{J_1(\omega R)}{\omega} f(x) \quad \text{see [1, formula 9.1.10]} \end{aligned}$$

where J_1 is a Bessel function of the first kind of order one. This implies that f satisfies Equation (6) if and only if

$$\omega R = 2J_1(\omega R).$$

Note that since $|J_1'(x)| < 1/2$ for all non-zero real x, the imaginary part of ω is non-zero. As in the previous section, we now consider functions of the form

$$f = \beta \cdot x e^{i\lambda\cdot x},$$

where $\beta \in C^2$. One may easily show that

$$\Delta^k f = (-\omega^2)^{k-1}(-\omega^2 f + 2ki\beta \cdot \lambda e^{i\lambda\cdot x}).$$

From Abrabowitz and Stegun[1, formula 9.1.10] we have

$$J_2(z) = (z/2)^2 \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{k!(k+2)!}.$$

We use this to obtain

$$\begin{aligned}
\int_{B_R(x)} f(t)dt &= 2\pi R \sum_{k=0}^{\infty} \frac{\Delta^k f(x)}{k!(k+1)!} \left(\frac{R}{2}\right)^{2k+1} \\
&= 2\pi R \sum_{k=0}^{\infty} \frac{(-\omega^2)^{k-1}(-\omega^2 f + 2ki\beta \cdot \lambda e^{i\lambda\cdot x})}{k!(k+1)!} \left(\frac{R}{2}\right)^{2k+1} \\
&= 2\pi R \frac{J_1(\omega R)}{\omega} f(x) + 4\pi R i\beta \cdot \lambda e^{i\lambda\cdot x} \sum_{k=0}^{\infty} \frac{(-\omega^2)^{k-1}}{k!(k+1)!} k \left(\frac{R}{2}\right)^{2k+1} \\
&= 2\pi R \frac{J_1(\omega R)}{\omega} f(x) + 2\pi R^2 i\beta \cdot \lambda e^{i\lambda\cdot x} J_2(\omega R)/\omega^2 \\
&= \pi R^2 f(x) + 2\pi R^2 i\beta \cdot \lambda e^{i\lambda\cdot x} J_2(\omega R)/\omega^2
\end{aligned}$$

since we have $\omega R = 2J_1(\omega R)$. Since $\omega \neq 0$ and the zeros of J_2 are real, we must have $\beta \cdot \lambda = 0$. We may now apply the remark to Theorem 2.5 to obtain

Theorem 4.1 *The space of continuous functions which satisfy Equation (6) is the closed linear span of the functions satisfying*

$$\Delta f + \omega^2 f = 0$$

where $\omega R = 2J_1(\omega R)$.

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