A NEW REPRESENTATION FOR LEGENDRE POLYNOMIALS

MARC CHAMBERLAND AND KAIQIAN ZHANG

ABSTRACT. By identifying the terms in the LU decomposition of an appropriate matrix, a new representation for Legendre polynomials is found.

In [1], Chamberland uses the LU decomposition of matrices, a tool typically used in numerical linear algebra, to discover and prove combinatorial identities. Specifically, take a highly structured square matrix, compute the LU decomposition, identify the terms in both L and U, and thus produce a conjectured sum formula. To see the patterns in L and U, one usually needs to consider an $n \times n$ matrix where n is sufficiently large. Sequence recognition is supported by using the On-Line Encyclopedia of Integer Sequences (`http://oeis.org/`) or the Maple package `gfun`.

The goal of this paper is to use the LU decomposition process to discover and prove a new representation for the Legendre polynomials. Identities and properties of these polynomials are ubiquitous in the literature[3]. A standard way to define the Legendre polynomials is with its Rodrigue's representation:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n] \tag{1}$$

where n is a natural number. Another approach is to generate these polynomials from the recurrence relationship

$$(n + 2)P_{n+2}(x) = (2n + 3)xP_{n+1}(x) - (n + 1)P_n(x) \tag{2}$$

coupled with $P_0(x) = 1$ and $P_1(x) = x$. Both of these characterizations play a role in the ensuing analysis.

Inspired by the Rodrigue representation (1), construct an $n \times n$ matrix M whose (i, j) entry is

$$M_{ij} = \frac{d^{i-1}}{dx^{i-1}} [(x^2 - 1)^{j-1}]$$

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The LU factorization, performed with Maple, produces the following when $n = 4$:

$$\begin{bmatrix} 1 & x^2-1 & (x^2-1)^2 & (x^2-1)^3 \\ 0 & 2x & 4x(x^2-1) & 6x(x^2-1)^2 \\ 0 & 2 & 12x^2-4 & 6(x^2-1)(5x^2-1) \\ 0 & 0 & 24x & 120x^3-72x \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/x & 1 & 0 \\ 0 & 0 & 3/x & 1 \end{bmatrix} \begin{bmatrix} 1 & x^2-1 & (x^2-1)^2 & (x^2-1)^3 \\ 0 & 2x & 4x(x^2-1) & 6x(x^2-1)^2 \\ 0 & 0 & 8x^2 & 24x^2(x^2-1) \\ 0 & 0 & 0 & 384x^4 \end{bmatrix}$$

By choosing larger values of n and looking for a pattern, one eventually conjectures forms for the (i, j) entry of both L and U:

$$L_{ij} = \begin{cases} \frac{(2i-2j)!\binom{i-1}{2i-2j}}{2^{i-j}(i-j)!}x^{j-i}, & i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$U_{ij} = \begin{cases} \frac{2^{i-1}(j-1)!}{(j-1)!}(x^2-1)^{j-i}x^{i-1}, & i \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

Since

$$M_{i,j} = \sum_{k=1}^{\min(i,j)} L_{i,k}U_{k,j},$$

this leaves us with the conjecture (after some simplification)

$$(3)\quad \frac{d^i}{dx^i}[(x^2-1)^j] = \sum_{k=0}^{\min(i,j)} \frac{j!(2i-2k)!}{(i-k)!(j-k)!}\binom{i}{2i-2k}(2x)^{2k-i}(x^2-1)j-k$$

Since our goal is to find a representation for Legendre polynomials, we are not interested in proving this formula in its full generality, but only in the special case $i = j$. Coupling this observation with Rodrigue's representation (1) suggests that we consider the polynomial expressions

$$f_j := \frac{1}{j!2^j}\sum_{k=0}^{j} \frac{j!(2j-2k)!}{((j-k)!)^2}\binom{j}{2j-2k}(2x)^{2k-j}(x^2-1)^{j-k}$$

It is possible, albeit cumbersome, to prove that f_j is the j^{th} Legendre polynomial by using known identities. However, this approach can be avoided by using Zeilberger's algorithm (see [2]) for combinatorial sums. Given a sum of hypergeometric type, this technique produces a recurrence relation satisfied by the sum. Using Maple's built-in command for Zeilberger's algorithm, one finds that

$$(j+2)f_{j+2}(x) = (2j+3)xP_{j+1}(x) - (j+1)P_j(x)$$

for all natural numbers j, the same recurrence as equation (2). It is easy to see that $f_0 = 1$ and $f_1 = x$, implying that the expression f_j is indeed the j^{th} Legendre polynomial, that is,

$$P_j(x) = \sum_{k=0}^{j} \frac{(2j-2k)!}{2^j((j-k)!)^2} \binom{j}{2j-2k} (2x)^{2k-j}(x^2-1)^{j-k} \tag{4}$$

This new expression can be compared to two similar well-known expressions [3]:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}(x+1)^k \tag{5}$$

and

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n-2k}{k} x^{n-2k} \tag{6}$$

The first formula is readily expanded around $x = \pm 1$, while the second formula expands around $x = 0$. The new formula (4) can be expanded around all three values.

References

[1] Chamberland, M. *Factored Matrices can generate Combinatorial Identities.* Electronic Journal of Combinatorics, **1**:1–9. 1994.

[2] Petkovsek, M., Wilf, H. and Zeilberger, D. *A=B*. AK Peters, Natick, MA, 1996.

[3] *Legendre polynomials, Wikipedia.* `http://mathworld.wolfram.com/LegendrePolynomial.html`

Department of Mathematics and Statistics, Grinnell College, Grinnell, IA 50112,

E-mail address: `chamberl@grinnell.edu`

Department of Mathematics and Statistics, Grinnell College, Grinnell, IA 50112,