WEAKENED MARKUS–YAMABE CONDITIONS FOR 2–DIMENSIONAL GLOBAL ASYMPTOTIC STABILITY

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Abstract. For a general 2–dimensional autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, it is difficult to find easily verifiable sufficient conditions guaranteeing global asymptotic stability of an equilibrium point. This paper considers three conditions which imply global asymptotic stability for a large class of systems, weakening the so–called Markus–Yamabe condition. The new conditions are: (1) the system admits a unique equilibrium point, (2) it is locally asymptotically stable, and (3) the trace of the Jacobian matrix of $\mathbf{f}$ is negative everywhere. We prove that under these three conditions global asymptotic stability is obtained when the components of $\mathbf{f}$ are polynomials of degree two or represent a Liénard system. However, we provide examples that global asymptotic stability is not obtained under these conditions for other classes of planar differential systems.

1. Introduction and statement of the main results

Since the time of Liapunov, it has become evident that finding conditions which guarantee global asymptotic stability of an equilibrium point in a differential system, even in two dimensions, is difficult. Liapunov's approach is probably the most wide–spread general method used, though constructing a Liapunov function usually requires ingenuity, experience, and some luck. For the 2–dimensional autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{1}$$

we seek a set of easily verifiable conditions which may give global asymptotic stability. A non–intuitive result to this end was proven in 1993, the so-called Markus–Yamabe conjecture in two dimensions (see [5], [7], [8]). This result shows that global asymptotic stability is obtained if the eigenvalues of the Jacobian matrix $D\mathbf{f}(\mathbf{x})$ have negative real part for all $\mathbf{x}$ in the plane. The aim of this paper is to weaken the

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Markus–Yamabe condition and still obtain global asymptotic stability for some classes of differential systems (1) in dimension 2.

The Markus–Yamabe condition is equivalent to having trace $D\mathbf{f} < 0$ and det $D\mathbf{f} > 0$ at every point. The trace condition itself is equivalent to having each region of finite area shrink under the flow, while the determinant condition has no known geometric interpretation. Several results (see [3], [10], [11], [6]) obtain global asymptotic stability by dropping the determinant condition, yet asking that an equilibrium point is unique, locally asymptotically stable, and adding a new condition in a neighborhood of infinity. The new requirements on the equilibrium point are reasonable since they are necessary for global asymptotic stability and relatively easy to check. Conditions near of infinity, however, may be more difficult to check, non-intuitive, and unnecessary. In this paper we deal with the following open problem.

Open Problem. *Suppose* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfies the following conditions:*

(C_1) $f(\mathbf{x}) = \mathbf{0}$ *if and only* $\mathbf{x} = \mathbf{p}$.
(C_2) *The equilibrium point* $\mathbf{p}$ *is locally asymptotically stable.*
(C_3) *Trace* $D\mathbf{f} < 0$ *for all* $\mathbf{x} \in \mathbb{R}^2$.

Then, when $\mathbf{p}$ *is globally asymptotically stable?*

If the components of $\mathbf{f}$ are polynomials of degree at most 2 and at least one of them has degree 2, then system (1) is called a *quadratic polynomial differential system* or simply a *quadratic system.*

Our main results are stated in the following theorem.

Theorem 1. *The following three statements hold.*

(a) *Every quadratic system satisfying assumptions (C_1)–(C_3) is globally asymptotically stable.*
(b) *Every Liénard system of the form*

$$\dot{x} = y - f(x), \qquad \dot{y} = -x, \tag{2}$$

with $f(x) = a_1 x + a_2 x^2 + \cdots + a_d x^d$ *and satisfying assumptions (C_1)–(C_3) is globally asymptotically stable.*
(c) *The differential system*

$$\dot{x} = -\frac{x(x+1)}{(1+y^2)^{3/2}}, \qquad \dot{y} = 4x + \frac{(2x-1)y}{\sqrt{1+y^2}}, \tag{3}$$

satisfies assumptions (C_1)–(C_3) and it is not globally asymptotically stable.

In Section 2 we prove statement (a) of Theorem 1, a non–trivial result since there are 111 different quadratic phase portraits with no limit cycles having a unique finite singular point (see [4]). Consequently, this shows that the Open Problem has a positive answer for the quadratic polynomial differential systems.

In Section 3 we show that the Open Problem also has a positive answer for Liénard systems with polynomial components; i.e. we prove statement (b) of Theorem 1.

Finally, in Section 4 we provide a negative answer to the Open Problem showing the existence of an algebraic vector field **f** which is non–rational and satisfies the three conditions of the Open Problem. In particular, the existence of this vector field proves statement (c) of Theorem 1.

Recently, the authors known that there are polynomial differential systems of degree 7 satisfying assumptions (C_1)–(C_3) and for which the system is not globally asymptotically stable, see [2].

Two questions which remain open are:

(1) *What is the largest family of planar differential systems for which the Open Problem has a positive answer?*
(2) *What is the maximum degree of polynomial differential systems for which the assumptions (C_1)–(C_3) imply that all the systems with that degree are globally asymptotically stable?*

2. Quadratic polynomial differential systems

If system (1) is linear, then condition (C_1) is met only if the Jacobian matrix at the origin has no zero eigenvalue. It is well–known that the only configurations in this case also satisfying conditions (C_2) and (C_3) are globally attracting nodes or foci. So, the Open Problem holds for linear systems.

The next proposition shows statement (a) of Theorem 1.

Proposition 2. *The only quadratic systems having a unique equilibrium point and satisfying the three conditions of the open problem (up to affine equivalence and time re–scaling) can be written as either*

$$\dot{x} = -x \ , \quad \dot{y} = -by - lx^2 \ , \quad b > 0 \ ;$$

or

$$\dot{x} = -x \ , \quad \dot{y} = -x - y - lx^2 \ .$$

Moreover, their equilibrium point is globally asymptotically stable.

For proving Proposition 2 we will use the next theorem providing the local phase portraits of semi–hyperbolic equilibrium points, for a proof of it see [1].

Theorem 3. *Let* $(0,0)$ *be an isolated equilibrium point of the system*

$$\dot{x} = F(x,y), \;\; \dot{y} = y + G(x,y)$$

where F *and* G *are analytic in a neighborhood of the origin and have expansions that begin with second degree terms in* x *and* y*. Let* $y = g(x)$ *be the solution of the equation* $y + G(x,y) = 0$ *in a neighborhood of* $(0,0)$*, and assume that the series expansion of the function* $F(x, g(x))$ *has the form* $a_m x^m + \cdots$ *, where* $m \geq 2$*,* $a_m \neq 0$*. Then*

(a) *If* m *is odd and* $a_m > 0$*, then* $(0,0)$ *is a topological node.*
(b) *If* m *is odd and* $a_m < 0$*, then* $(0,0)$ *is a topological saddle.*
(c) *If* m *is even, then* $(0,0)$ *is a saddle–node.*

Proof of Proposition 2: A key point in our proof is the following classification of quadratic systems having a unique equilibrium point up to affine equivalence and scaling the time variable (see [4] for more details):

(I.e): $\dot{x} = y - x^2 + xy, \;\; \dot{y} = ax + by + Q_2(x,y)$ with $a \neq 0$;
(I.s): $\dot{x} = y - x^2 + xy, \;\; \dot{y} = by + Q_2(x,y)$ with $b \neq 0$;
(I.h): $\dot{x} = y - x^2 + xy, \;\; \dot{y} = Q_2(x,y)$;
(II.e): $\dot{x} = xy, \;\; \dot{y} = ax + by + Q_2(x,y)$ with $a \neq 0$;
(II.s): $\dot{x} = xy, \;\; \dot{y} = by + Q_2(x,y)$ with $b \neq 0$;
(II.h): $\dot{x} = xy, \;\; \dot{y} = Q_2(x,y)$;
(III.e): $\dot{x} = y + x^2, \;\; \dot{y} = \pm x + y + Q_2(x,y)$ with $n = 0$ and, either $m \neq 0$ and $(l-b)^2 \pm 4m < 0$, or $m = 0$ and $l = b$;
(III.s): $\dot{x} = y + x^2, \;\; \dot{y} = y + Q_2(x,y)$ with either $n \neq 0$ and $m^2 - 4n(l-1) < 0$, or $n \neq 0$, $m = 0$ and $l = 1$, or $n = 0$, $m \neq 0$ and $l = 1$, or $n = m = 0$ and $l \neq 1$;
(III.h): $\dot{x} = y + x^2, \;\; \dot{y} = Q_2(x,y)$ with either $n \neq 0$ and $m^2 - 4nl < 0$, or $n \neq 0$, $m = l = 0$, or $n = l = 0$, $m \neq 0$, or $n = m = 0$ and $l \neq 0$;
(IV.e): $\dot{x} = y, \;\; \dot{y} = \pm x + by + Q_2(x,y)$ with $b \geq 0$ and $l = 0$;
(IV.s): $\dot{x} = y, \;\; \dot{y} = y + Q_2(x,y)$ with $l \neq 0$;
(IV.h): $\dot{x} = y, \;\; \dot{y} = Q_2(x,y)$ with $l \neq 0$;
(V.e): $\dot{x} = x^2 - 1, \;\; \dot{y} = d + by + lx^2 + mxy$ with $m \neq 0$ and $d + l \neq 0$;
(V.s): $\dot{x} = x^2 - 1, \;\; \dot{y} = d + ax + by + lx^2 + mxy + y^2$ with $(b+m)^2 - 4(d+a+l) = 0$ and $(b-m)^2 - 4(d-a+l) < 0$;
(VII.s): $\dot{x} = x^2, \;\; \dot{y} = y + Q_2(x,y)$ with $n = 0$;
(VII.h): $\dot{x} = x^2, \;\; \dot{y} = x + Q_2(x,y)$ with $n = 1$;
(VIII.e1): $\dot{x} = x, \;\; \dot{y} = by + Q_2(x,y)$ with $b \neq 0$ and $n = 0$;
(VIII.e2): $\dot{x} = x, \;\; \dot{y} = x + y + Q_2(x,y)$ with $n = 0$;
(VIII.s): $\dot{x} = x, \;\; \dot{y} = Q_2(x,y)$ with $n \neq 0$;

Homogeneous: $\dot{x} = P_2(x, y),\ \dot{y} = Q_2(x, y)$;
where $P_2(x, y) = Lx^2 + Mxy + Ny^2$ and $Q_2(x, y) = lx^2 + mxy + ny^2$.

For proving the proposition we must go through each sub–case in the above classification. The expression "Trace" will refer to the trace of the Jacobian matrix of the system.

(I.e) $\dot{x} = y - x^2 + xy$, $\dot{y} = ax + by + Q_2(x, y)$ with $a \neq 0$. Since Trace $= -2x + y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 1/2$ and $n = -1/2$. This reduces the system to $\dot{x} = y - x^2 + xy$, $\dot{y} = ax + by + lx^2 + 2xy - y^2/2$. If (x_0, y_0) is an equilibrium point, the first equation yields $y_0 = x_0^2/(x_0 + 1)$, which in conjunction with the second equation implies

$$x_0 \left(a + \frac{bx_0}{x_0 + 1} + lx_0 + \frac{2x_0^2}{x_0 + 1} - \frac{x_0^3}{2(x_0 + 1)^2} \right) = 0. \tag{4}$$

If $x_0 = 0$, then $y_0 = 0$. If $x_0 \neq 0$, then equation (4) may be written as

$$(3 - 2l)x_0^3 + 2(a + b + 2l + 2)x_0^2 + 2(2a + b + l)x_0 + 2a = 0.$$

To satisfy condition (C_1), we require $l = 3/2$, leaving

$$2(a + b + 5)x_0^2 + (4a + 2b + 3)x_0 + 2a = 0. \tag{5}$$

Condition (C_2) implies $a < 0$ since this case assumes that $a \neq 0$ and the determinant of the Jacobian at the origin equals $-a$. Solving for x_0 in equation (5) yields a discriminant equalling

$$(4a + 2b + 3)^2 - 16a(a + b + 5) = 4(b + 3/2)^2 - 56a > 0,$$

hence condition (C_1) is violated unless $a + b + 5 = 0$. Equation (5) would further require that $4a + 2b + 3 = 0$. Solving for a and b yield $a = 7/2$, contradicting the earlier observation that $a < 0$. Therefore, this case cannot satisfy all the conditions.

(I.s) $\dot{x} = y - x^2 + xy$, $\dot{y} = by + Q_2(x, y)$ with $b \neq 0$. Since Trace $= -2x + y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 2$ and $n = -1/2$, yielding $\dot{x} = y - x^2 + xy$, $\dot{y} = by + lx^2 + 2xy - y^2/2$. Making the change of variables $X = y - bx$ and $Y = y$ transforms the system into

$$\begin{aligned} \dot{X} &= (l + b)(Y - X)^2/b^2 + (2 - b)Y(Y - X)/b - Y^2/2, \\ \dot{Y} &= bY + l(Y - X)^2/b^2 + 2Y(Y - X)/b - Y^2/2. \end{aligned}$$

Solving $bY + l(Y - X)^2/b^2 + 2Y(Y - X)/b - Y^2/2 = 0$ for Y in a neighborhood of $(0, 0)$ yields

$$Y = -lX^2/b^3 + \cdots,$$

so Theorem 3 implies $(0, 0)$ is a saddle–node, contradicting condition (C_2).

(I.h) $\dot{x} = y - x^2 + xy$, $\dot{y} = Q_2(x,y)$. Since Trace $= -2x + y + mx + 2ny$, condition (C_3) is never satisfied.
(II.e) $\dot{x} = xy$, $\dot{y} = ax + by + Q_2(x,y)$ with $a \neq 0$. Since Trace $= y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$, $n = -1/2$ and $b \neq 0$. Since both $(0,0)$ and $(0,2b)$ are equilibria, condition (C_1) is satisfied only if $b = 0$, a contradiction.
(II.s) $\dot{x} = xy$, $\dot{y} = by + Q_2(x,y)$ with $b \neq 0$. Since Trace $= y + b + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$ and $n = -1/2$. Since both $(0,0)$ and $(0,2b)$ are equilibria, condition (C_1) is satisfied only if $b = 0$, a contradiction.
(II.h) $\dot{x} = xy$, $\dot{y} = Q_2(x,y)$. Since Trace $= y + mx + 2ny$, condition (C_3) is never satisfied.
(III.e.i) $\dot{x} = y + x^2$, $\dot{y} = \pm x + by + lx^2 + mxy$ with $m \neq 0$ and $(l-b)^2 \pm 4m < 0$. Since Trace $= 2x + b + mx$, condition (C_3) is satisfied only if $m = -2$ and $b < 0$, therefore we must choose the plus sign, leaving the system $\dot{x} = y + x^2$, $\dot{y} = x + by + lx^2 - 2xy$. At the equilibrium point $(0,0)$, the Jacobian's determinant is -1, contradicting condition (C_2).
(III.e.ii) $\dot{x} = y + x^2$, $\dot{y} = \pm x + by + bx^2$. Since Trace $= 2x + b$, condition (C_3) is never satisfied.
(III.s) $\dot{x} = y + x^2$, $\dot{y} = y + Q_2(x,y)$ with either $n \neq 0$ and $m^2 - 4n(l-1) < 0$, or $n \neq 0$, $m = 0$ and $l = 1$, or $n = 0$, $m \neq 0$ and $l = 1$, or $n = m = 0$ and $l \neq 1$. Since Trace $= 2x + 1 + mx + 2ny$, condition (C_3) is satisfied only if $m = -2$ and $n = 0$. Only one of the four possible sub–cases fits, implying $l = 1$. Since Trace $= 1$, we time–reverse the system to yield $\dot{x} = -y - x^2$, $\dot{y} = -y - x^2 + 2xy$. Making the change of variables $X = x - y$ and $Y = y$ transforms the system into $\dot{X} = 2XY - 2Y^2$, $\dot{Y} = Y + X^2 - Y^2$. Solving $Y + X^2 - Y^2 = 0$ for Y in a neighborhood of $(0,0)$ yields

$$Y = \frac{1 - \sqrt{1 + 4X^2}}{2} = -X^2 + \cdots ,$$

so Theorem 3 implies $(0,0)$ is a saddle–node, contradicting condition (C_2).
(III.h) $\dot{x} = y + x^2$, $\dot{y} = Q_2(x,y)$. Since Trace $= 2x + mx + 2ny$, condition (C_3) is never satisfied.
(IV.e) $\dot{x} = y$, $\dot{y} = \pm x + by + Q_2(x,y)$ with $b \geq 0$ and $l = 0$. Since Trace $= b + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$ and $n = 0$. This reduces the system to a linear one, with which we have already dealt.
(IV.s) $\dot{x} = y$, $\dot{y} = y + Q_2(x,y)$ with $l \neq 0$. Since Trace $= 1 + mx + 2ny$, condition (C_3) is satisfied only if $m = 0$, $n = 0$, and the system is

time–reversed, leaving us with $\dot{x} = -y$, $\dot{y} = -y - lx^2$. Making the change of variables $X = y - x$ and $Y = y$ transforms the system into $\dot{X} = l(Y - X)^2$, $\dot{Y} = Y + l(Y - X)^2$. Solving $Y + l(Y - X)^2 = 0$ for Y in a neighborhood of $(0, 0)$ yields

$$Y = \frac{2lX - 1 + \sqrt{(2lX-1)^2 - 4l^2X^2}}{2l} = -lX^2 + \cdots ,$$

so Theorem 3 implies $(0, 0)$ is a saddle–node, contradicting condition (C_2).
(IV.h) $\dot{x} = y$, $\dot{y} = Q_2(x, y)$. Since Trace $= mx + 2ny$, condition (C_3) is never satisfied.
(V.e) $\dot{x} = x^2 - 1$, $\dot{y} = d + by + lx^2 + mxy$ with $m \neq 0$ and $d + l \neq 0$. Since Trace $= 2x + b + mx$, condition (C_3) is satisfied only if $m = -2$. To meet condition (C_1), we must take $b = \pm 2$, with the x–coordinate of the equilibrium point equaling $x = \mp 1$. The determinant of the Jacobian at this point is negative in both cases, hence condition (C_2) is violated.
(V.s) $\dot{x} = x^2 - 1$, $\dot{y} = d + ax + by + lx^2 + mxy + y^2$. Since Trace $= 2x + b + mx + 2y$, condition (C_3) is never satisfied.
(VII.s) $\dot{x} = x^2$, $\dot{y} = y + Q_2(x, y)$ with $n = 0$. Since Trace $= 2x + 1 + mx$, condition (C_3) is satisfied only if $m = -2$ and the system is time–reversed, leaving us with $\dot{x} = -x^2$, $\dot{y} = -y - lx^2 + 2xy$. Solving $-y - lx^2 + 2xy = 0$ for y in a neighborhood of $(0, 0)$ yields

$$y = \frac{-lx^2}{1 - 2x} = lx^2 + \cdots$$

so Theorem 3 implies that $(0, 0)$ is a saddle–node, contradicting condition (C_2).
(VII.h) $\dot{x} = x^2$, $\dot{y} = x + Q_2(x, y)$ with $n = 1$. Since Trace $= 2x + mx + 2y$, condition (C_3) is never satisfied.
(VIII.e1) $\dot{x} = x$, $\dot{y} = by + Q_2(x, y)$ with $b \neq 0$ and $n = 0$. Since Trace $= 1 + b + mx$, condition (C_3) is satisfied only if $m = 0$. The unique equilibrium point $(0, 0)$ satisfies condition (C_2) only if $b > 0$ and the system is time–reversed, giving $\dot{x} = -x$, $\dot{y} = -by - lx^2$. One may show directly that this new system is globally asymptotically stable at $(0, 0)$. First, $x = Ce^{-t}$ for some constant C. Using this to solve for y yields

$$y = \begin{cases} De^{-bt} - (lC^2e^{-2t})/(b-2), & b \neq 2 \\ De^{-2t} - lC^2te^{-2t}, & b = 2 \end{cases}$$

for some constant D. Conditions (C_1)–(C_3) are all satisfied if and only if $b > 0$.

(VIII.e2) $\dot{x} = x$, $\dot{y} = x + y + Q_2(x, y)$ with $n = 0$. Since Trace $= 2 + mx$, condition (C_3) is satisfied only if $m = 0$ and the system is time–reversed, yielding $\dot{x} = -x$, $\dot{y} = -x - y - lx^2$. One may show directly that this new system is globally asymptotically stable at $(0, 0)$. First, $x = Ce^{-t}$ for some constant C. Using this to solve for y yields $y = De^{-t} - Cte^{-t} + lC^2e^{-2t}$ for some constant D. The conditions (C_1)–(C_3) are all satisfied.

(VIII.s) $\dot{x} = x$, $\dot{y} = Q_2(x, y)$ with $n \neq 0$. Since Trace $= 1 + mx + 2ny$, condition (C_3) is satisfied only if $n = 0$ which contradicts the assumption of this case.

Homogeneous quadratic: $\dot{x} = P_2(x, y)$, $\dot{y} = Q_2(x, y)$. Since Trace $= 2Lx + My + mx + 2ny$, condition (C_3) is never satisfied. ∎

3. Liénard systems

In this section we prove statement (b) of Theorem 1. A study (see [9]) of such systems in a neighborhood of infinity on the Poincaré sphere forms the backbone of the proof. There are four possibilities; see Figure 1.

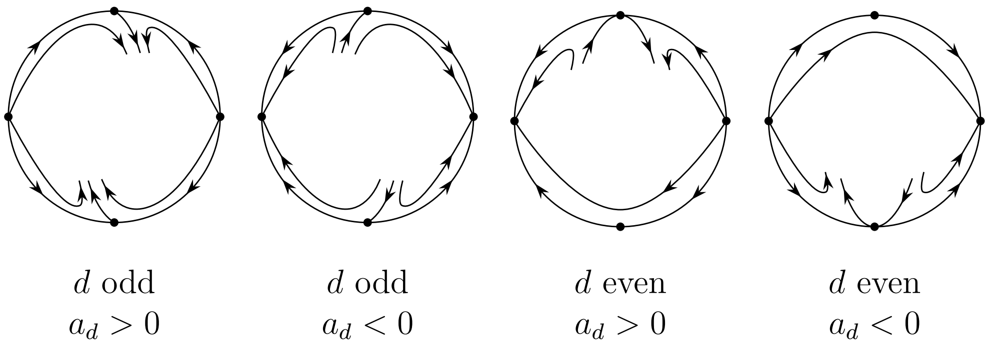

Figure 1. Poincaré spheres for Liénard systems.

The proof follows straightforwardly. Condition (C_3) implies d is odd, and also, by the Bendixson Theorem (see for instance [13]) that no periodic orbits exist. Conditions (C_1) and (C_2), with the Poincaré–Bendixson Theorem (see for instanced [12]), imply that the first picture of Figure 1 is the only possibility, and that $(0, 0)$ is globally asymptotically stable.

4. A negative answer to the open problem

In this section we prove statement (c) of Theorem 1. It is easy to check that the unique equilibrium point of system (3) is $(0, 0)$, so (C_1)

holds. Since the eigenvalues of the linear part of this system at $(0,0)$ are both equal to -1, $(0,0)$ is a stable node. Therefore, (C_2) is satisfied. The trace at any point (x,y) is $-2/(1+y^2)^{3/2}$. Hence, the system also satisfies (C_3). Finally, the equilibrium point $(0,0)$ is not globally asymptotically stable since the line $x=-1$ is invariant.

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