Jacobian Conjectures: Injectivity and Dynamical Systems

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Jacobian Conjectures ask (global) questions pertaining to maps and flows based on information from their (local) Jacobian matrices. First consider the

Markus-Yamabe Conjecture(1960): Suppose $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and one considers the differential equation $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ where the eigenvalues λ of $\mathbf{f}'(\mathbf{x})$ have $\Re\lambda < 0$ for all λ at all points $\mathbf{x}$ and $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, then $\mathbf{x} = 0$ is globally attracting.

Status: When $n = 2$, the conjecture is true (see [7], [8], and [9] for proofs). When $n \geq 3$ the conjecture is false; Bernat and Llibre[2] produced a four-dimensional real-analytic counter-example with a periodic orbit, while Cima *et al.*[6] present a simple three-dimensional counter-example with divergent trajectories.

For polynomial maps satisfying the Markus-Yamabe conditions, Olech[10] made connections between global attractivity and injectivity, motivating the following conjecture:

Weak Polynomial Markus-Yamabe Conjecture: If $\mathbf{f}$ is a polynomial map satisfying the Markus-Yamabe conditions, then $\mathbf{f}$ is injective.

Status: True when $n = 2$ (using Olech's work and two-dimensional proof of the Markus-Yamabe Conjecture), open for $n \geq 3$.

Loosening these conditions a bit leads to the

Real Jacobian Conjecture on $\mathbb{R}^n$: Every polynomial map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det \mathbf{F}'(\mathbf{x}) \neq 0$ is injective.

Status: Pinchuk[11] found a counter-example in $n = 2$ involving polynomials of degrees 10 and 25. This connects to the classical

(Keller) Jacobian Conjecture on $\mathbb{R}^n$ (1939): Every polynomial map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det \mathbf{F}'(\mathbf{x}) \equiv 1$ is a bijective map with a polynomial inverse.

Status: Open, even for $n = 2$.

This problem is usually posed over the field $\mathbb{C}^n$. Important reductions to the Jacobian Conjecture have shown that it is sufficient to prove that all *cubic-homogeneous* maps are injective ([12] and [3] deal with inkectivity, Bass *et al.*[1] and Yagzhev[13] deal with cubic-homogeneous maps). A map $\mathbf{F}$ is in *cubic-homogeneous* form if $\mathbf{F}(\mathbf{x}) = \mathbf{x} - \mathbf{H}(\mathbf{x})$ where $\mathbf{H}(t\mathbf{x}) = t^3\mathbf{H}(\mathbf{x})$ for all $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. It should be noted that the Jacobian Conjecture is false for non-polynomial maps: consider the map $(u, v) = (\sqrt{2}e^{x/2}\cos{(ye^{-x})}, \sqrt{2}e^{x/2}\sin{(ye^{-x})})$ whose Jacobian determinant is identically one, yet it is not injective.

Note that cubic-homogeneous maps have Jacobian matrices whose eigenvalues are always one at all points. Such matrices are dubbed *unipotent*. A very recent study [4] has proven that all C^1 two-dimensional maps

$\mathbf{f}$ with unipotent Jacobians are invertible and of the form

$$\mathbf{f}(x,y) = (x + b\phi(ax+by) + c, \quad y - \phi(ax+by) + d)$$

for some constants a, b, c, d and a C^1 function ϕ. Similar results for higher dimensions are unknown.

These results have motivated (see [4, 5]) the following so-called

Chamberland Conjecture: Every C^1 map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ such that the eigenvalues of $\mathbf{F}'(\mathbf{x})$ are uniformly bounded away from zero is injective.

Status: Open, even for $n = 2$.

In light of these results and conjectures, two fundamental questions may be posed:

Question 1: Why are results often different in dimension two compared to higher dimensions?

Question 2: Why are some results dependent on the function $\mathbf{f}$ being a polynomial while others are not?

References

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