A MOUNTAIN PASS TO THE JACOBIAN CONJECTURE

Marc Chamberland and Gary Meisters

June 7, 1997

Abstract. This paper presents an approach to injectivity theorems via the Mountain Pass Lemma and raises a new open question. The main result of this paper (Theorem 1.1) is proved by means of the Mountain Pass Lemma and states that if all the eigenvalues of $F'(\boldsymbol{x})F'(\boldsymbol{x})^T$ are bounded away from zero for all $\boldsymbol{x} \in \mathbb{R}^n$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a class $\mathcal{C}^1$ map, then F is injective. This was discovered in a joint attempt by the authors to prove a stronger result conjectured by the first author: Namely, that a sufficient condition for injectivity of class $\mathcal{C}^1$ maps F of $\mathbb{R}^n$ into itself is that all the eigenvalues of $F'(\boldsymbol{x})$ are bounded away from zero on $\mathbb{R}^n$. This is stated as Conjecture 2.1. If true, it would imply (via *Reduction-of-Degree*) *injectivity of polynomial maps* $F : \mathbb{R}^n \to \mathbb{R}^n$ *satisfying the hypothesis*, $\det F'(\boldsymbol{x}) \equiv 1$, of the celebrated Jacobian Conjecture (**JC**) of Ott-Heinrich Keller. The paper ends with several examples to illustrate a variety of cases and known counterexamples to some natural questions.

1991 *Mathematics Subject Classification.* Primary 14A25; Secondary 14E09.
Key words and phrases. Injectivity, $\mathcal{C}^1$-maps, polynomial maps, Jacobian Conjecture, Mountain Pass.
The first author was partially supported by an NSERC Post-Doctoral Fellowship.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

1. A Sufficient Condition for Injectivity

The purpose of this paper is to show that the celebrated Mountain Pass Lemma can be used to prove a new sufficient condition for *injectivity* of $\mathcal{C}^1$ maps $F : \mathbb{R}^n \to \mathbb{R}^n$. We discovered this new injectivity theorem while trying to prove a different injectivity conjecture of the first author, stated below as Conjecture 2.1, whose truth would clearly imply injectivity of *polynomial* maps $F : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the hypothesis, $\det F'(\boldsymbol{x}) \equiv 1$, of the Jacobian Conjecture of Ott-Heinrich Keller [16]. We regard elements $\boldsymbol{x}$ of $\mathbb{R}^n$ as *column* vectors $[x_1, \dots, x_n]^T$ and write $F'(\boldsymbol{x})$ for the Jacobian matrix of F at $\boldsymbol{x}$, whose entry in its i^{th}-row and j^{th}-column is $[F'(\boldsymbol{x})]_{ij} = \partial F_i(\boldsymbol{x})/\partial x_j$.

Theorem 1.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map. Suppose there exists an $\epsilon > 0$ such that $|\mu| \geq \epsilon$ for all the eigenvalues μ of $F'(\boldsymbol{x})F'(\boldsymbol{x})^T$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Then F is injective.*

To prove this we use the **Mountain Pass Lemma** due to Ambrosetti & Rabinowitz [1]. The statement of it given below as Lemma 1.1 is taken from [4]. Let E be a Banach space and $h : E \to \mathbb{R}$ a function which might satisfy any of the following conditions.

C_1. **The Palais-Smale Compactness Condition at one value $a \in \mathbb{R}$:**

$$(PS)_a \quad \begin{cases} \text{Every sequence } \{\boldsymbol{x}_k\} \text{ in } E, \text{ such that } h(\boldsymbol{x}_k) \to a \text{ and } \|h'(\boldsymbol{x}_k)\| \to 0, \\ \text{has a convergent subsequence.} \end{cases}$$

C_2. **The Palais-Smale Compactness Condition:**

$$(PS) \quad \begin{cases} \text{We say that } h \text{ satisfies } (PS) \text{ if } (PS)_a \text{ holds for every } a \in \mathbb{R}. \\ \text{This condition was originally introduced by Palais \& Smale [23].} \end{cases}$$

C_3. **The Mountain Pass Condition:**

$$(MP) \quad \begin{cases} \text{There is an open neighborhood } U \text{ of } \mathbf{0} \text{ and some point } \boldsymbol{x}_0 \notin \overline{U} \\ \text{such that } \max\{h(\mathbf{0}), h(\boldsymbol{x}_0)\} < m := \inf\{h(\boldsymbol{x}) : \boldsymbol{x} \in \partial U\}. \\ \text{Let } \mathcal{A} \text{ denote the family of all continuous paths } g : [0,1] \to E \\ \text{joining } \mathbf{0} \text{ to } \boldsymbol{x}_0, \text{ and put } c := \inf_{g \in \mathcal{A}} \max_{t \in [0,1]} h(g(t)). \text{ Clearly } c \geq m. \end{cases}$$

Lemma 1.1. (The Standard MPL) *Let $h : E \to \mathbb{R}$ be a $\mathcal{C}^1$ function satisfying (MP). Then there exists a sequence $\{\boldsymbol{x}_k\}$ in E such that*

$$h(\boldsymbol{x}_k) \to c \quad \textit{and} \quad \|h'(\boldsymbol{x}_k)\| \to 0.$$

If h also satisfies $(PS)_c$ with c defined as in (MP), then c is a **critical value** *of h: That is, for some $\boldsymbol{x}_c \in E$, $h(\boldsymbol{x}_c) = c$ and $h'(\boldsymbol{x}_c) = \mathbf{0}^T = (0, \dots, 0)$.*

For more about Critical Point Theory and The Mountain Pass Lemma see [1,4,23,26,28] and the references cited there. To prove Theorem 1.1 we apply the Mountain Pass Lemma 1.1, in the finite-dimensional real Banach space $E = \mathbb{R}^n$, to an appropriately defined real-valued function $h : \mathbb{R}^n \to \mathbb{R}$ associated with the given $\mathcal{C}^1$ mapping $F : \mathbb{R}^n \to \mathbb{R}^n$.

The 1-dimensional example $F(x) = \arctan(x)$ suffices to show that the condition in Theorem 1.1 is *not necessary* for the injectivity of F.

2. Proof of Theorem 1.1 & A New Open Question

The Proof. If F is not injective, then $F(\boldsymbol{x}_1) = F(\boldsymbol{x}_2)$ for two distinct vectors $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n$. Let $G(\boldsymbol{x}) := F(\boldsymbol{x}+\boldsymbol{x}_1) - F(\boldsymbol{x}_1)$, $\boldsymbol{x}_0 = \boldsymbol{x}_2 - \boldsymbol{x}_1$, and $h(\boldsymbol{x}) := \frac{1}{2}G(\boldsymbol{x})^T G(\boldsymbol{x})$. Then h is real-valued on $\mathbb{R}^n$ and $h(\mathbf{0}) = h(\boldsymbol{x}_0) = 0$. Since $h'(\boldsymbol{x}) = G(\boldsymbol{x})^T G'(\boldsymbol{x})$, and $G'(\boldsymbol{x})$ is invertible for all $\boldsymbol{x} \in \mathbb{R}^n$, every point $\boldsymbol{x}_c$ at which $h'(\boldsymbol{x}_c) = \mathbf{0}^T$ must satisfy $G(\boldsymbol{x}_c) = \mathbf{0}$, and so also $h(\boldsymbol{x}_c) = 0$. Furthermore, $\boldsymbol{x} = \mathbf{0}$ *is an isolated zero of* h. To see this we argue as follows. By applying the Classical Mean Value Theorem, $f(1) - f(0) = f'(\theta)$ for some $0 < \theta < 1$, to $f(t) := G_i(t\boldsymbol{x})$ for each component $G_i(\boldsymbol{x})$ of $G(\boldsymbol{x})$ and each nonzero $\boldsymbol{x} \in \mathbb{R}^n$, we obtain

$$G_i(\boldsymbol{x}) = {G_i}'(\theta_i(\boldsymbol{x})\boldsymbol{x})\boldsymbol{x} \quad \text{where} \quad 0 < \theta_i(\boldsymbol{x}) < 1 \quad \text{for} \quad i = 1, \dots, n. \tag{1}$$

Then the $n \times n$ matrix $A(\boldsymbol{x})$, defined by $A(\mathbf{0}) = G'(\mathbf{0})$ and

$$A(\boldsymbol{x}) = \begin{pmatrix} {G_1}'(\theta_1(\boldsymbol{x})\boldsymbol{x}) \\ \vdots \\ {G_n}'(\theta_n(\boldsymbol{x})\boldsymbol{x}) \end{pmatrix}, \quad \text{when} \quad \boldsymbol{x} \neq \mathbf{0}, \tag{2}$$

satisfies the equation

$$G(\boldsymbol{x}) = A(\boldsymbol{x})\boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^n. \tag{3}$$

Now define $B : \mathbb{R}^n \times \overset{n}{\cdots} \times \mathbb{R}^n \to \mathcal{M}_{n\times n}(\mathbb{R})$ by

$$B(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) = \begin{pmatrix} {G_1}'(\boldsymbol{x}_1) \\ \vdots \\ {G_n}'(\boldsymbol{x}_n) \end{pmatrix}. \tag{4}$$

Then B is continuous, $B(\boldsymbol{x}, \dots, \boldsymbol{x}) = G'(\boldsymbol{x})$, and $B(\theta_1(\boldsymbol{x})\boldsymbol{x}, \dots, \theta_n(\boldsymbol{x})\boldsymbol{x}) = A(\boldsymbol{x})$. Since

$$\det B(\mathbf{0}, \dots, \mathbf{0}) = \det G'(\mathbf{0}) \neq 0, \tag{5}$$

there exists a real number $r > 0$ such that $\det B(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \neq 0$ when $\sum_{i=1}^n \|\boldsymbol{x}_i\|^2 \leq r^2$. Therefore, $\det A(\boldsymbol{x}) \neq 0$ if $\|\boldsymbol{x}\|^2 \leq r^2/n$. So, by (3), $G(\boldsymbol{x}) = \mathbf{0}$ inside the ball $\|\boldsymbol{x}\| \leq r/\sqrt{n}$ only at $\boldsymbol{x} = \mathbf{0}$, which is thus *an isolated zero of* $h(\boldsymbol{x})$. Also $\|\boldsymbol{x}_0\| > r/\sqrt{n}$ since $G(\boldsymbol{x}_0) = \mathbf{0}$.

Thus the function $h(\boldsymbol{x}) := \frac{1}{2}G(\boldsymbol{x})^T G(\boldsymbol{x})$ satisfies the Mountain Pass Condition (MP), with $U = \{\, \boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| < r/\sqrt{n} \,\}$, $m = \min\{\, h(\boldsymbol{x}) : \|\boldsymbol{x}\| = r/\sqrt{n} \,\}$, and $\boldsymbol{x}_0 = \boldsymbol{x}_2 - \boldsymbol{x}_1$. The infimum m is positive because the sphere $S_{r/\sqrt{n}} = \{\, \boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| = r/\sqrt{n} \,\} = \partial U$ is compact, h is continuous, and h is not zero on $S_{r/\sqrt{n}}$. Now the c in (MP) cannot be a critical value of h because if $h'(\boldsymbol{x}_c) = \mathbf{0}^T$ then, as we noted above, $h(\boldsymbol{x}_c) = 0$ too; so we would obtain the contradiction $0 < m \leq c = h(\boldsymbol{x}_c) = 0$. Thus, by Lemma 1.1, the compactness condition $(\mathrm{PS})_c$ does not hold for our function $h(\boldsymbol{x}) := \frac{1}{2}G(\boldsymbol{x})^T G(\boldsymbol{x})$. Therefore there is a sequence $\{\boldsymbol{x}_k\} \in \mathbb{R}^n$ such that

(i) $\lim_{k\to\infty} \|\boldsymbol{x}_k\| = \infty$;
(ii) $\lim_{k\to\infty} h(\boldsymbol{x}_k) = c \geq m > 0$; i.e., $\lim_{k\to\infty} \|G(\boldsymbol{x}_k)\| = \sqrt{2c} > 0$; and
(iii) $\lim_{k\to\infty} h'(\boldsymbol{x}_k) = \lim_{k\to\infty} G(\boldsymbol{x}_k)^T G'(\boldsymbol{x}_k) = \mathbf{0}^T$.

If μ_1 denotes the **minimum eigenvalue** of a hermitian matrix A, one has the well-known inequality $\mu_1 = \inf_{\boldsymbol{y}\neq 0}(\boldsymbol{y}^T A\boldsymbol{y})/(\boldsymbol{y}^T\boldsymbol{y})$. That is, the minimum eigenvalue of a symmetric matrix A is the infimum of the **Rayleigh Quotient** $\rho(\boldsymbol{y}) := (\boldsymbol{y}^T A\boldsymbol{y})/(\boldsymbol{y}^T\boldsymbol{y})$. See [22]. Applying this with $\boldsymbol{y} = G(\boldsymbol{x})$ and $A = G'(\boldsymbol{x})G'(\boldsymbol{x})^T$ (a positive definite hermitian matrix), we obtain at $\boldsymbol{x} = \boldsymbol{x}_k$ for all $k \geq 1$,

$$0 < \mu_1(\boldsymbol{x}_k) \leq G(\boldsymbol{x}_k)^T G'(\boldsymbol{x}_k)G'(\boldsymbol{x}_k)^T G(\boldsymbol{x}_k)/G(\boldsymbol{x}_k)^T G(\boldsymbol{x}_k), \quad \text{for } G(\boldsymbol{x}_k) \neq \mathbf{0}. \tag{6}$$

Since the numerator $\|G'(\boldsymbol{x}_k)^T G(\boldsymbol{x}_k)\|^2$ in the right-hand-side of (6) tends to 0 by (iii) while the denominator tends to $2c > 0$ by (ii), we obtain $\mu_1(\boldsymbol{x}_k) \to 0$. Since this contradicts the hypothesis of Theorem 1.1 that *all eigenvalues of* $F'(\boldsymbol{x})F'(\boldsymbol{x})^T$ *are bounded away from zero*, it follows that F must be injective. This completes the proof of Theorem 1.1. □

Compare the statement of Theorem 1.1 with that of the following open conjecture.

Conjecture 2.1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$ map. Suppose there exists an $\epsilon > 0$ such that $| \lambda | \geq \epsilon$ for all the eigenvalues λ of $F'(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Then F is injective.*

The example $F(x) = \arctan(x)$, mentioned after Theorem 1.1, also shows that the condition in Conjecture 2.1 is *not necessary* for the injectivity of F. In order for the argument used in the proof of Theorem 1.1 to prove Conjecture 2.1, we would need to show that $\lim_{k\to\infty} \lambda_1(\boldsymbol{x}_k) = 0$, where

$$\lambda_1(\boldsymbol{x}_k) := \min\{\, | \lambda | : \lambda \text{ an eigenvalue of } G'(\boldsymbol{x}_k)\,\}.$$

However, $\mu_1 \to 0$ does *not* imply that $\lambda_1 \to 0$, as the following example on $\mathbb{R}^2$ shows. Let $(u, v) = F(x, y) = (ax - y^3, by)$, $ab \neq 0$. This is a polyomorphism of $\mathbb{R}^2$ whose Jacobian matrix $F'(x, y)$ has eigenvalues $\{a, b\}$ (hence bounded away from zero); but, nevertheless, the minimum eigenvalue μ_1 of the positive definite symmetric matrix $F'(x, y)F'(x, y)^T$ *does tend to zero* as $y \to \infty$. Thus $\mu_1 \to 0$ does *not* imply that $\lambda_1 \to 0$; so no contradiction. *Why should the eigenvalue of $G'(\boldsymbol{x}_k)$ with minimum absolute value $| \lambda(\boldsymbol{x}_k) |$ go to zero as $k \to \infty$?* Of course there is the well-known inequality $| \lambda(\boldsymbol{x}_k) | < \| G'(\boldsymbol{x}_k) \|$; *but there is no reason to believe that $\| G'(\boldsymbol{x}_k) \| \to 0$ as $k \to \infty$.* However, one might think that we could apply the above mentioned Rayleigh Quotient idea to the (non-symmetric) matrix $A = G'(\boldsymbol{x}_k)$ because (at that point in the proof) we know by (iii) that $\lim_{k\to\infty} h'(\boldsymbol{x}_k) = \lim_{k\to\infty} G(\boldsymbol{x}_k)^T G'(\boldsymbol{x}_k) = \mathbf{0}^T$, so that

$$\lim_{k\to\infty} G(\boldsymbol{x}_k)^T G'(\boldsymbol{x}_k)G(\boldsymbol{x}_k)/G(\boldsymbol{x}_k)^T G(\boldsymbol{x}_k) = 0. \tag{7}$$

But this Rayleigh Quotient for $G'(\boldsymbol{x}_k)$ *does not majorize* $\lambda_1(\boldsymbol{x}_k)$, as can be seen for the example $F(x, y) = (ax - y^3, by)$, with $ab \neq 0$, so no contradiction arises.

It is interesting to compare our Theorem 1.1 with the [1906] **Theorem of Hadamard** on the global injectivity of proper maps [14]: A $\mathcal{C}^1$ map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism if and only if f is proper and the Jacobian determinant of f, $\det f'(\boldsymbol{x})$, never vanishes. Recall [see, e.g., Bourbaki's **General Topology** Part 1, Chapter I, § 10] that a continuous map f is said to be **proper** if $f^{-1}(K)$ is compact whenever K is compact; and a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is proper if and only if $\|\boldsymbol{x}\| \to \infty$ implies $\|f(\boldsymbol{x})\| \to \infty$. Thus a proper map satisfies (PS). The proof of Theorem 1.1 essentially argues that, supposing f is not injective, the condition (PS) is violated, which leads to the contradiction.

Note that the eigenvalue-hypothesis of Theorem 1.1 can be expressed in another way. For real square matrices A and B, $\|A\|^2 = \|A^T A\| = r(A^T A)$, the spectral radius of $A^T A$; and the eigenvalues of B and B^{-1} are reciprocals of each other. Also $\det AA^T = (\det A)^2$ so the product of the eigenvalues of AA^T is the square of the product of the eigenvalues of A. Thus the eigenvalues of $F'(\boldsymbol{x})F'(\boldsymbol{x})^T$ are bounded from zero on $\mathbb{R}^n$ if and only if $\det F'(x)$ never vanishes on $\mathbb{R}^n$ and $\|F'(\boldsymbol{x})^{-1}\|$ is bounded on $\mathbb{R}^n$. From this one can see that our Theorem 1.1 is related to results in the earlier papers of Plastock [25], Smyth & Xavier [31], and Rabier [27].

3. The Jacobian Conjecture

Let k be a field of characteristic zero, and k^n the vector space over k of column n-tuples of elements of k. Call $F : k^n \to k^n$ a **polynomial map** if each of its components $F_i(\boldsymbol{x})$ belongs to the polynomial ring $k^{[n]} = k[\boldsymbol{x}] = k[x_1, x_2, \ldots, x_n]$; a **Keller map** if also $\det F'(\boldsymbol{x}) \equiv$ constant $\neq 0$; and a **polyomorphism** if, in addition, F is bijective with polynomial inverse. **The Jacobian Conjecture**, expressed in this terminology, is that the following question, raised in 1939 by Ott-Heinrich Keller [16], has an affirmative answer.

Conjecture 3.1. (k-JC) *Is every Keller map of k^n a polyomorphism of k^n?*

For the exact form and context of Keller's original question see [16]; and for more general versions and a survey of its history and many other references see [2,12,17,18,19,20,21,32]. In this paper, we are only concerned with the two cases: $k = \mathbb{R}$ and $k = \mathbb{C}$. In all that follows in this paper, we may (and do) assume without loss of generality that Keller maps are normalized to satisfy $\det F'(\boldsymbol{x}) \equiv 1$. Proofs of $\mathbb{R}$-JC and of $\mathbb{C}$-JC can be based on the following two reductions, which are independent of each other. First, for $\mathbb{C}$-JC it suffices to prove *injectivity* [3,30]:

Reduction 3.1. (Injectivity Suffices.) *Injective polynomial maps of $\mathbb{R}^n$ or $\mathbb{C}^n$ into itself are surjective. Bijective polynomial maps of $\mathbb{C}^n$ into itself have polynomial inverses.*

Using this reduction, it follows immediately from the elementary mean-value formula

$$F(\boldsymbol{x}) - F(\boldsymbol{y}) = F'\left(\frac{\boldsymbol{x}+\boldsymbol{y}}{2}\right)(\boldsymbol{x}-\boldsymbol{y})$$

for polynomial maps $F : k^n \to k^n$ of *degree two*, that (1) $\mathbb{C}$-JC is true in all dimensions for all polynomial maps of *degree two*; and (2) all *quadratic* polynomial maps of $\mathbb{R}^n$ into itself with nonvanishing Jacobians are bijective. (Pinchuk's counterexample [24] has degree 25.)

The second reduction [2,33] is called **reduction of degree** and states that *by increasing dimension (the number of variables) it is possible to reduce the degree of the polynomial map F to three with no term of degree two.* We may also assume $F(0) = 0$ and $F'(0) = I$.

Reduction 3.2. (Reduction of Degree.) *Keller maps $F : k^n \to k^n$, of every degree and in every dimension n, are injective if and only if, in every dimension n, Keller maps of the special "cubic-homogeneous" form $F(\boldsymbol{x}) = \boldsymbol{x} - H(\boldsymbol{x})$, where $H(t\boldsymbol{x}) = t^3 H(\boldsymbol{x})$ for all $t \in k$ and for all $\boldsymbol{x} \in k^n$, are also injective. For these, Keller's condition* $\det F'(\boldsymbol{x}) \equiv 1$ *is* *equivalent to the condition that the Jacobian matrix* $H'(\boldsymbol{x})$ *is nilpotent* $\forall \boldsymbol{x} \in k^n$*: Indeed, if* $\det(I - H'(\boldsymbol{x})) \equiv 1$*, then,* $\forall \lambda > 0$*,* $\det(\lambda I - H'(\boldsymbol{x})) \equiv \lambda^n \det\left(I - H'\left(\boldsymbol{x}/\sqrt{\lambda}\right)\right) \equiv \lambda^n$*; so the characteristic polynomial of* $H'(\boldsymbol{x})$ *is* λ^n*, and* $H'(\boldsymbol{x})^n \equiv O$ *by the Cayley-Hamilton Theorem. Consideration of the minimum polynomial of* $H'(\boldsymbol{x})$ *gets the converse.*

4. Keller Maps on $\mathbb{R}^n$ and $\mathbb{C}^n$

What would the truth of Conjecture 2.1 entail for polynomial maps of $\mathbb{R}^n$ & $\mathbb{C}^n$?

Theorem 4.1. *Truth of Conjecture 2.1 implies Keller maps of* $\mathbb{R}^n$ *are injective.*

Proof. By the Reduction Theorem 3.2 it suffices to prove that Keller maps of the special "cubic-homogeneous" form $F(\boldsymbol{x}) = \boldsymbol{x} - H(\boldsymbol{x})$, where $H(t\boldsymbol{x}) = t^3 H(\boldsymbol{x})$ for all $t \in k$ and for all $\boldsymbol{x} \in k^n$, are injective in every dimension $n \geq 2$. But such maps, as noted, satisfy the condition that the Jacobian matrix $H'(\boldsymbol{x})$ is *nilpotent* for every $\boldsymbol{x}$ in k^n: $H'(\boldsymbol{x})^n \equiv O$. And it follows from this condition that all eigenvalues of $F'(\boldsymbol{x}) = I - H'(\boldsymbol{x})$ are $+1$, because $\forall \lambda < 1$, $\det(F'(\boldsymbol{x}) - \lambda I) \equiv (1-\lambda)^n \det\left(I - H'\left(\boldsymbol{x}/\sqrt{1-\lambda}\right)\right) \equiv (1-\lambda)^n$. So now by Conjecture 2.1 these cubic-homogeneous Keller maps are indeed injective. □

Whether or not bijective Keller maps of $\mathbb{R}^n$ have *polynomial* inverses remains to be seen.

Theorem 4.2. *Truth of Conjecture 2.1 implies Keller maps of* $\mathbb{C}^n$ *are polyomorphisms.*

Proof. Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a Keller map of $\mathbb{C}^n$. Define the associated real map $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by $\tilde{F} = (\Re F_1, \Im F_1, \ldots, \Re F_n, \Im F_n)$. Now $\det \tilde{F}' = |\det F'|^2$, so $\det \tilde{F}'$ is a nonzero constant if and only if $\det F'$ is a nonzero constant. Consequently $\tilde{F} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a Keller map of $\mathbb{R}^{2n}$, and hence injective by Theorem 4.1. But one easily sees that F is injective if and only if $\tilde{F}$ is injective. It now follows from the Reduction Theorem 3.1 of Białynicki-Birula & Rosenlicht [3], recently given a more elementary proof by Rudin [30], that F is bijective with a polynomial inverse (i.e., a polyomorphism). □

Here are a few examples to illustrate the situation to date, as far as we know it.

5. Recent Polynomial Examples are Consistent with Conjecture 2.1

Example 5.1. *Cubic-homogeneous maps* $F(\boldsymbol{x}) := \boldsymbol{x} - H(\boldsymbol{x})$*, where* $H(t\boldsymbol{x}) = t^3 H(\boldsymbol{x})$ *and* $H'(\boldsymbol{x})^n \equiv \mathbf{O}$*. All known examples of Keller maps of this type are polyomorphisms. The Jacobian Conjecture is true by virtue of the Reduction 3.2 iff no polymap of this type, in any dimension, fails to be injective. Each polymap of this form has all eigenvalues of its Jacobian matrix equal to* $+1$*, and thus satisfies the hypotheses of Conjecture 2.1.*

Example 5.2. *Injective cubic-homogeneous maps need not be polynomially linearizable* [10].

$$F = (\,X_1 + X_4(X_3X_1 + X_4X_2), X_2 - X_3(X_3X_1 + X_4X_2), X_3 + {X_4}^3, X_4\,)^T$$

This example has the property that its dilations sF *are* **not polynomially linearizable***; but they are* **analytically linearizable***. That is, there is an s-family* $\phi_s(\boldsymbol{x})$*, holomorphic in* $\boldsymbol{x}$ *but not polynomial in* $\boldsymbol{x}$*, such that* $\phi_s \circ sF \circ {\phi_s}^{-1} = sX$*.* [9,15].

Example 5.3. *Injective cubic-homogeneous maps need not be analytically linearizable* [11].

$$F = (\, X_1 + X_4(X_3X_1 + X_4X_2)^2, X_2 - X_3(X_3X_1 + X_4X_2)^2, X_3 + {X_4}^3, X_4 \,)^T$$

Dilations sF of this map are **not even analytically linearizable**. *But sF could perhaps be linearizable for some s-family of bijections $\phi_s(\boldsymbol{x})$.*

Example 5.4. *A polynomial counterexample in $\mathbb{R}^3$ to the* Markus-Yamabe-Conjecture [7].

$$F = (\, -X_1 + X_3(X_1 + X_2X_3)^2, -X_2 - (X_1 + X_2X_3)^2, -X_3 \,)^T$$

Zero is not a global attractor of the differential equation $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$ even though $F(0) = 0$, F is injective, and $\forall \boldsymbol{x} \in \mathbb{R}^3$ all eigenvalues of $F'(\boldsymbol{x})$ have negative real parts (all are -1). *Nevertheless, the system $\dot{\boldsymbol{x}} = F(\boldsymbol{x})$ has the solution $\boldsymbol{x}(t) = (\, 18e^t, -12e^{2t}, e^{-t} \,) \not\to \mathbf{0}$.*

6. Non-Injective Examples

Example 6.1. **Pinchuk**'*s polynomial map $P : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies* $\det P'(\boldsymbol{x}) > 0$ *for all $\boldsymbol{x} \in \mathbb{R}^2$, but is neither injective nor surjective: Indeed, $P(1,0) = P(-1,-2) = (0,-1)$; and $P(\mathbb{R}^2) = \mathbb{R}^2 \setminus \{two\ points\}$. Pinchuk's polymap $P(\boldsymbol{x}) = P(x,y) = ((p(x,y), q(x,y))$ is defined, $\forall \boldsymbol{x} = (x,y)^T$, by the equations $t := xy - 1$, $h := t(xt+1)$, $f := (h+1)(xt+1)^2/x$, $p(x,y) := f + h$, $Q := -t^2 - 6th(h+1)$, $u := 170fh + 91h^2 + 195fh^2 + 69h^3 + 75h^3 f + (75/4)h^4$, $q(x,y) := Q - u$; so* $\det P'(x,y) = \partial(p,q)/\partial(x,y) = t^2 + (t + (13+15h)f)^2 + f^2$. *Therefore, along the curve $xy = 1$, we have $t = 0$ so the Jacobian reduces to* $\det P'(x,y) = 170y^2$ *which clearly tends to zero as $y \to 0$ and $x \to \infty$. Consequently, not both eigenvalues of $P'(\boldsymbol{x})$ are bounded away from zero; so Pinchuk's non-injective polynomial map does not satisfy the hypotheses of Conjecture 2.1.* See [6,8,24].

Example 6.2. *A* **non-injective analytic Keller map** $F : \mathbb{C}^2 \to \mathbb{C}^2$, *which also maps $\mathbb{R}^2$ into $\mathbb{R}^2$, is given by $F(x,y) := (u(x,y), v(x,y))^T$ where*

$$F \begin{cases} u(x,y) := \sqrt{2}e^{x/2}\cos(ye^{-x}) \\ v(x,y) := \sqrt{2}e^{x/2}\sin(ye^{-x}). \end{cases}$$

Indeed, $F(0, y + 2k\pi) = F(0,y)$, so F is not injective even though $\det F'(\boldsymbol{x}) \equiv 1$. *However, not both eigenvalues of $F'(\boldsymbol{x})$ are bounded away from zero.*

Example 6.3. *The following* **non-injective analytic Samuelson map** $F : \mathbb{R}^2 \to \mathbb{R}^2$ *was given by Gale & Nikaidô in 1965* [13]. *Let $F(x,y) := (f(x,y), g(x,y))^T$ where*

$$F \begin{cases} f(x,y) := e^{2x} - y^2 + 3 \\ g(x,y) := 4ye^{2x} - y^3. \end{cases}$$

A $\mathcal{C}^1$ self-map of real n-space is called a **Samuelson map** *if the leading principle minors $\mu_1, \ldots, \mu_n$ of its Jacobian matrix vanish nowhere. These maps are named after the economist and Nobel laureate, Paul A. Samuelson, who suggested in 1953 that such maps should be univalent (injective).*

For Gale & Nikaidô's map given here, $F(0,2) = F(0,-2) = (0,0)$, *so* F *is not injective even though* $\mu_1 = 2e^{2x}$ *and* $\mu_2 = \det F'(\boldsymbol{x}) = e^{2x}(8e^{2x} + 10y^2)$ *are everywhere positive. Note that the eigenvalues of* $F'(\boldsymbol{x})$ *are not bounded from zero because, along each line* $y = c$, $\det F'(\boldsymbol{x}) \to 0$ *as* $x \to -\infty$. *Also note that* $F(\mathbb{R}^2) = \mathbb{R}^2 \setminus \{(3,0)\}$. *However, everywhere defined rational Samuelson maps are injective.* See Campbell [5].

7. Examples with Eigenvalues Bounded from Zero

Every *polynomial* map of the form $F(x) = x - H(x)$, with nilpotent Jacobian $H'(x)$ of its homogeneous part $H(x)$, has the property that *all the eigenvalues of its Jacobian matrix* $F'(x)$ are equal to 1 (and hence are bounded away from zero). If it were *known* to be true that all such *polynomial* maps are *injective*, then the Jacobian Conjecture would be known to be true as well (by reduction of degree). The Conjecture 2.1 claims more: That any $\mathcal{C}^1$-map F must be injective if the eigenvalues of its Jacobian matrix $F'(x)$ are bounded away from zero. In the absence of counterexamples, it is interesting to see some nontrivial (and *non-polynomial*) examples where it is true: Thus the statement of Conjecture 2.1 is at least true for *some* non-polynomial analytic maps.

Example 7.1. *An* **injective analytic map** *with* **eigenvalues bounded from zero**. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by* $F(\boldsymbol{x}) = F(x,y) := (u(x,y), v(x,y))^T$ *where* $\boldsymbol{x} = (x,y)^T$ *and*

$$F \begin{cases} u(x,y) := x - e^{\cos^2 y}, \text{ and} \\ v(x,y) := 2y + e^{\cos^2 y}. \end{cases}$$

Its eigenvalues are $\{1, \lambda(y)\}$ *where* $\lambda(y) = 2 - e^{\cos^2 y} \sin 2y \geq 0.1540508716\ldots$. *This can be easily seen as follows:* $\lambda(y)$ *is a continuous function of period* π, *so its maximum and minimum must occur at points where its derivative* $\lambda'(y)$ *is zero. This happens when* $(\cos 2y)^2 + 2\cos 2y - 1 = 0$, *or when* $\cos 2y = \sqrt{2} - 1$, *or* $y = \pm 0.5718588702\ldots$. *The minimum occurs at the positive value and the maximum occurs at the negative value: Thus* $\lambda_{min} = 0.1540508716\ldots$ and $\lambda_{max} = 3.8459491284\ldots$. *Since* $\|F(\boldsymbol{x})\| \to \infty$ *whenever* $\|\boldsymbol{x}\| \to \infty$, *and* $\det F'(\boldsymbol{x})$ *never vanishes, Hadamard's theorem for proper local diffeomorphisms tells us that this mapping* F *is a global diffeomorphism of* $\mathbb{R}^2$.

Example 7.2. *Another injective analytic map with* **eigenvalues bounded from zero**. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be given by* $F(\boldsymbol{x}) = F(x,y) := (u(x,y), v(x,y))^T$ *where* $\boldsymbol{x} = (x,y)^T$ *and*

$$F \begin{cases} u(x,y) := 2x + \cos x \cos y, \text{ and} \\ v(x,y) := 2y - \cos x \cos y. \end{cases}$$

Its eigenvalues are $\{2, \lambda(x,y)\}$ *where* $\lambda(x,y) = 2 - \sin(x-y) \geq 1$. *Again* $\|F(\boldsymbol{x})\| \to \infty$ *whenever* $\|\boldsymbol{x}\| \to \infty$, *and* $\det F'(\boldsymbol{x})$ *never vanishes, so Hadamard's theorem for proper local diffeomorphisms tells us that this mapping* F *is a global diffeomorphism of* $\mathbb{R}^2$.

Example 7.3. Non-injective rational maps with eigenvalues bounded from zero. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given, for any integer $p \geq 2$, by*

$$F \begin{cases} u(x,y) := x - \dfrac{p}{p+1}(\dfrac{x}{y})^{p+1}, \; \text{and} \\ v(x,y) := y - (\dfrac{x}{y})^p. \end{cases}$$

Its eigenvalues are $\{1,1\}$ and $F(6,3) = F(12,-3) = (4,1)$ when $p = 1$.

Example 7.4. Open: *An integer $n \geq 2$ and a* **non-injective** *$\mathcal{C}^1$-map $F : \mathbb{R}^n \to \mathbb{R}^n$ with* **eigenvalues bounded from zero**. *This would be a counterexample to the Conjecture 2.1. If no such counterexample exists, then the Jacobian Conjecture is true.*

8. Acknowledgements

During the preparation of this paper the first author was partially supported by a two-year Post-Doctoral Fellowship from the **Natural Sciences and Engineering Research Council of Canada**, which is gratefully acknowledged. Since January 1997, his income was generously supplemented by a group of six Professors at McMaster University. The second author benefited from a discussion with his colleague Tom Shores which led him to the example $F(x,y) = (ax - y^3, by)$ mentioned twice on page 4. The authors thank L. A. Campbell, Arno van den Essen, Czesław Olech, and the referee for remarks which led to some improvements in the exposition.

References

1. A. Ambrosetti & P. Rabinowitz, *Dual Variational Methods in Critical Point Theory and Applications*, J. Funct. Anal. **14** (1973), 349–381.
2. H. Bass, E. H. Connell & D. Wright, *The Jacobian Conjecture: Reduction of Degree and Formal Expansion of the Inverse*, Bull. Amer. Math. Soc. **7** (1982), 287–330.
3. A. Białynicki-Birula & M. Rosenlicht, *Injective morphisms of real algebraic varieties*, Proc. Amer. Math. Soc. **13** (1962), 200–203.
4. H. Brezis & L. Nirenberg, *Remarks on Finding Critical Points*, Communications on Pure and Applied Mathematics **44** (1991), 939–963.
5. L. A. Campbell, *Rational Samuelson maps are univalent*, J. Pure Appl. Algebra **92** (1994), 227–240.
6. L. A. Campbell, *Partial Properness and Real Planar Maps*, Appl. Math. Lett. **9, No. 5** (1996), 99–105.
7. A. Cima, A. van den Essen, A. Gasull, E. Hubbers, and F. Mañosas, *A polynomial counterexample to the Markus-Yamabe Conjecture on $\mathbb{R}^3$*, Univ. of Nijmegen, The Netherlands, Mathematics Dept. **Report No. 9551** (Nov. 1995). The final blow to the Markus-Yamabe Conjecture in dimensions $n > 2$; to appear in *Adv. in Math.*
8. A. van den Essen via B. A. Coomes, *Details of Pinchuk's degree(10,25) counterexample to the Strong Real Jacobian Conjecture*, Personal communication from Essen via Coomes June 3, 1994.
9. Bo Deng, *Analytic Conjugation, Global Attractor, and the Jacobian Conjecture*, TEX preprint, Spring 1995. Characterizes analytic maps which are globally analytically conjugate to their linear part at a fixed point. See also Rosay & Rudin [29].
10. A. van den Essen, *Counterexample to a conjecture of Meisters*, Univ. of Nijmegen, The Netherlands, Mathematics Dept. **Report 9441** (Oct. 1994). The Conjugation Conjecture for $F = X - H$ is false.
11. A. van den Essen & E. Hubbers, *Chaotic Polyomorphisms: Counterexamples to Several Conjectures*, Adv. in Appl. Math. **18, no. 3** (April 1997), 382–388. Nijmegen Report **9549** (November 1995).

12. A. van den Essen, editor, *Automorphisms of Affine Spaces*, Proceedings of a Conference on Polynomial Automorphisms, at the Princess Beach Resort Casino, Willemstad, Curaçao, The Netherlands Antilles, July 4–8, 1994. **Kluwer Academic Publishers** 1995, xix + 243 pp. ISBN 0-7923-3523-6.
13. D. Gale & H. Nikaidô, *The Jacobian matrix and global univalence*, Math. Ann. **159** (1965), 81–93.
14. W. B. Gordon, *On the Diffeomorphisms of Euclidean Space*, Amer. Math. Monthly **79** (1972), 755–759; see also *Addendum*, vol. **80** No. 6 (1973) 674–675; and *Application*, vol. **84** No. 1 (1977) 28–29.
15. G. Gorni & G. Zampieri, *On cubic-linear polynomial mappings*, UDMI/43/96/RR (September 1996), 21 pages, Udine, Italy; Nederl. Akad. Wetensch. Indag. Math. (1997), to appear.
16. O.-H. Keller, *Ganze Cremona Transformationen*, Monatshefte für Mathematik und Physik **47** (1939), 299–306; Items 6 & 7 in the table on page 301.
17. G. H. Meisters, *Jacobian problems in differential equations and algebraic geometry*, Rocky Mountain J. Math. **12** (1982), 679–705.
18. G. H. Meisters, *Inverting polynomial maps of n-space by solving differential equations*, in *Delay and Differential Equations*, Proceedings in Honor of George Seifert, Ames, Iowa, October 18–19, 1991. Eds. Fink/Miller/Kliemann **World Scientific,** Singapore · River Edge NJ · London · Hong Kong 1992, pp. 107–166. ISBN 981-02-0891-X.
19. G. H. Meisters, *Wanted: A Bad Matrix*, Amer. Math. Monthly **102** (1995), 546–550.
20. G. H. Meisters, *PolyMapBib 1997, 30 pp*, World Wide Web, http://www.math.unl.edu/~gmeister/.
21. G. H. Meisters, *A Biography of the Markus-Yamabe Conjecture*, in *Aspects of Mathematics: Algebra, Geometry and Several Complex Variables* **Proc. of Conf. June 10–13, 1996, at HKU** (to appear).
22. Noble & Daniel, *Applied Linear Algebra*, Prentice-Hall, 1977, 2^{nd} ed., p. 431, Thm 11.7.
23. R. S. Palais & S. Smale, *A generalized Morse Theory*, Bull. Amer Math Soc. **70** (1964), 165–171.
24. S. I. Pinchuk, *Counterexample to the Strong Real Jacobian Conjecture*, Math. Z. **217** (1994), 1–4.
25. R. Plastock, *Homeomorphisms between Banach spaces*, Trans. Amer. Math. Soc. **200** (1974), 169–183.
26. P. Pucci & J. Serrin, *The Structure of the Critical Set in the Mountain Pass Theorem*, Trans. Amer. Math. Soc. **299** (1987), 115–132.
27. P. J. Rabier, *On Global Diffeomorphisms of Euclidean Space*, Nonlinear Anal. **21** (1993), 925–947.
28. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS # 65, Amer. Math. Soc. 1986, for the Conference Board of the Mathematical Sciences, Expository Lectures of CBMS Regional Conference held at the Univ. of Miami, January 9–13, 1984.
29. J.-P. Rosay & W. Rudin, *Holomorphic maps from* $\mathbb{C}^n$ *to* $\mathbb{C}^n$, Trans. Amer. Math. Soc. **310** (1988), 47–86. Compare Theorem in the Appendix with Bo Ding [9].
30. W. Rudin, *Injective Polynomial Maps are Automorphisms*, Amer. Math. Monthly **102** (1995), 540–543.
31. B. Smyth & F. Xavier, *Injectivity of Local Diffeomorphisms from Nearly Spectral Conditions*, J. Differential Equations **130** (1996), 406–414.
32. D. Wright, *On the Jacobian Conjecture*, Illinois J. Math. **25** (1981), 423–440.
33. A. V. Yagzhev, *Keller's Problem*, Siberian Math. J. **21** (1981), 747–754.

Dept Mathematics & Statistics, McMaster Univ, Hamilton, Ontario, Canada, L8S 4K1
New Address: Dept Mathematics & Computer Science, Grinnell College, IA 50112, USA
E-mail address: marc@daniel.math.mcmaster.ca

Dept Mathematics and Statistics, University of Nebraska-Lincoln, NE 68588–0323, USA
E-mail address: gmeister@math.unl.edu