Diffeomorphic Real-Analytic Maps and the Jacobian Conjecture

Marc Chamberland
Department of Mathematics and Computer Science
Grinnell College, Grinnell, IA, 50112, U.S.A.
E-mail: chamberl@math.grin.edu.

Dedicated to Lloyd Jackson on the occasion of his 75th Birthday

Abstract. It is shown that if the real-analytic map $\mathbf{f}(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}^2$ has a Jacobian matrix whose eigenvalues are always both one, then the map is a diffeomorphism. An explicit form of the inverse given. The proof relies on a result which says that the only global solutions to the quasi-linear partial differential equation $(\cos u)u_x - (\sin u)u_y = 0$ are constant functions. The main result is put into a context concerning the Jacobian Conjecture.

AMS (MOS) subject classification : 14E09, 34A26, 34C05, 35F05.
Key Words. Jacobian Conjecture, polymorphisms, injectivity, Markus-Yamabe Conjecture.

1 Introduction

The Jacobian Conjecture is a long-standing open problem which has linked many ideas in algebra and analysis. We let $\mathbf{F}'(\mathbf{x})$ denote the Jacobian of the function $\mathbf{F}$ at $\mathbf{x}$. Limiting ourselves to the field $\mathbb{R}^n$, we have the

Jacobian Conjecture on $\mathbb{R}^n$: Is every polynomial map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det \mathbf{F}'(\mathbf{x}) \equiv 1$ a bijective map with a polynomial inverse?

This problem is usually posed over the field $\mathbb{C}^n$. Up to the writing of this paper, this conjecture has remained open, even in the case $n = 2$. This problem is not to be confused with the related

Real Jacobian Conjecture on $\mathbb{R}^n$: Is every polynomial map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\det \mathbf{F}'(\mathbf{x}) \neq 0$ injective?

This conjecture was proven false by Pinchuk[14]. The counter-example is with $n = 2$ and involves polynomials of degrees 10 and 25. More thorough discussions concerning the Jacobian Conjecture may be found in (for example) [1], [10], [17], and [18]. A thorough bibliography concerning polynomial maps and the Jacobian Conjecture is maintained by Gary Hosler Meisters[11].

Limiting ourselves to polynomial maps $\mathbf{F}$ with $\det \mathbf{F}'(\mathbf{x}) \equiv 1$, there are two important reductions to the Jacobian Conjecture.

Reduction 1: If the polynomial map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is injective, then it is also surjective.

Proofs may be found in [15] and [2]. A map $\mathbf{F}$ is in *cubic-homogeneous* form if $\mathbf{F}(\mathbf{x}) = \mathbf{x} - \mathbf{H}(\mathbf{x})$ where $\mathbf{H}(t\mathbf{x}) = t^3\mathbf{H}(\mathbf{x})$ for all $t \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

Reduction 2: The maps $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ are injective in every dimension n and every degree if and only if every such map of the cubic-homogeneous form is injective.

The second reduction was obtained by Bass, Connell and Wright[1] and Yagzhev[19]. Of relevance to this discussion is that the cubic-homogeneous polynomials have Jacobian matrices whose eigenvalues are all one at all points $\mathbf{x} \in \mathbb{R}^n$ (since $\mathbf{H}'(\mathbf{x})$ must be nilpotent).

In light of these reductions, many authors (see the forementioned references) have proved injectivity in a limited number of cases by considering the cubic-homogeneous polynomials. The main result of this paper takes a different approach.

Theorem 1.1 *Suppose* $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ *is real-analytic and* $\mathbf{f}'(x, y)$ *has both eigenvalues equal to one for all* $(x, y) \in \mathbb{R}^2$*. Then* $\mathbf{f}$ *is a diffeomorphism. Specifically, if*

$$\mathbf{f}(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u - c\sin\theta - (\cos\theta)h((\sin\theta)u + (\cos\theta)v - c) \\ v - c\cos\theta + (\sin\theta)h((\sin\theta)u + (\cos\theta)v - c) \end{pmatrix}$$

for some constants θ*,* c *and a real-analytic function* h*. Moreover, if* $\mathbf{f}$ *is a polynomial, then its inverse is also a polynomial.*

The proof depends on an interesting result concerning a quasi-linear partial differential equation. Section 2 is devoted to this result while Section 3 gives the proof of Theorem 1.1.

2 A Supporting Theorem

Theorem 2.1 *Let* $u : \mathbb{R}^2 \to \mathbb{R}$ *be a solution to*

$$(\cos u)u_x - (\sin u)u_y = 0. \tag{1}$$

Then u *is a constant function.*

Proof: Suppose $u = u(x, y)$ is a global solution to (1). The Method of Characteristics for quasi-linear partial differential equations implies that u is constant along solution curves of the system

$$\dot{x} = \cos u \tag{2}$$

$$\dot{y} = -\sin u \tag{3}$$

where x and y are functions of t, thus the base characteristics (orbits of the system) are straight lines. It is impossible for two orbits to cross because of uniqueness and the lack of critical points, hence all the orbits are parallel lines. This implies u must take the form

$$u(x,y) = k(x\cos\alpha + y\sin\alpha)$$

for some constant α and real-analytic function k. Substituting this into (1) gives

$$\begin{aligned} 0 &= (\cos u)(k')\cos\alpha - (\sin u)(k')\sin\alpha \\ &= k'\cos(u+\alpha). \end{aligned}$$

Both cases $k' \equiv 0$ and $\cos(u+\alpha) \equiv 0$ imply u is a constant function. □

Remark 2.1 The method of characteristics showed that the base characteristics are straight lines. This may be shown by considering the curvature of systems of the form

$$\dot{x} = \cos u(x,y) \tag{4}$$

$$\dot{y} = -\sin u(x,y) \tag{5}$$

for any smooth function u. One may show that for the system represented by

$$\dot{x} = f(x,y)$$

$$\dot{y} = g(x,y)$$

the curvature of the orbit passing through the non-equilibrium point (x,y) is given by

$$H(x,y) = \frac{f^2 g_x + (g_y - f_x)fg - f_y g^2}{(f^2+g^2)^{3/2}}.$$

The curvature of the orbits of a system was recently used by Chamberland[3] and Garcia *et al.*[7] to obtain results concerning global asymptotic stability in certain planar systems related to the Markus-Yamabe conjecture. Since the speed of the system (4)–(5) is always one, the curvature of the orbits of this system at (x, y) is

$$\begin{aligned} H(x,y) &= (\cos u)^2(-(\cos u)u_x) + [-(\cos u)u_y + (\sin u)u_x]\cos u(-\sin u) \\ &\quad - (-(\sin u)u_y)(\sin u)^2 \\ &= -(\cos u)u_x + (\sin u)u_y \\ &= 0 \quad \text{for all } x, y. \end{aligned}$$

This forces orbits of the system to lie on straight lines.

Remark 2.2 One may rewrite (1) as

$$\frac{\partial}{\partial x}\sin u + \frac{\partial}{\partial y}\cos u = 0. \tag{6}$$

By considering the system

$$\dot{x} = \sin u(x, y)$$

$$\dot{y} = \cos u(x, y)$$

equation (6) implies the divergence of this flow is identically zero. In contrast to Bendixson's Criteria for the elimination of periodic orbits, it has been suggested (see Perko[13, p.246]) that such a condition may imply a center exists. This is impossible for the flow under consideration since it has no equilibrium points. Theorem 2.1 indicates that not only is there no center, but all the orbits are parallel straight lines. This flow is orthogonal to the flow (2)–(3) used in the proof of the Theorem 2.1.

Remark 2.3 Imitating the previous remark, define the vector function

$$\mathbf{v} := (v_1, v_2) = (\sin u, \cos u)$$

to give

$$\nabla \cdot \mathbf{v} = 0, \quad |\mathbf{v}| = 1$$

Interpreted in a physical sense, Theorem 2.1 says that an imcompressible fluid which always moves with a non-zero constant speed must be a rigid translation.

Remark 2.4 Interpreting equation (6) as a condition for exactness, this forces the existence of a function $v = v(x, y)$ such that

$$v_x = \sin u, \quad v_y = -\cos u$$

which implies

$$v_x^2 + v_y^2 = 1$$

This is known as the *Eiconal equation*, or the equation of geometrical optics, which is used to describe wave fronts for a wave with constant speed (see Garabedian[6, pp.40–44]). To obtain a global solution, we would require that there are no caustics, that is, places where the wave front intersects itself. Theorem 2.1 confirms one's intuition that this may only occur when the wave front is a line. The avoidance of caustics alludes to the fact that having motion in straight lines (local result) is not sufficient in itself to give the desired result; we additionally require this condition to hold everywhere (globally).

3 Proof of Main Theorem

Now we prove Theorem 1.1.

Proof: By Schur's Theorem of matrix analysis (see, for example, [16, p.308]), there exists functions $A = A(x, y)$ and $\theta = \theta(x, y)$ such that

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & A \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} 1 + A(\cos\theta)(\sin\theta) & A(\cos\theta)^2 \\ -A(\sin\theta)^2 & 1 - A(\cos\theta)(\sin\theta) \end{bmatrix} \tag{8}$$

Since u and v are real-analytic, this forces A and θ to be real-analytic. Since $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$, we have

$$\frac{\partial}{\partial y}(A(\cos\theta)(\sin\theta)) = \frac{\partial}{\partial x}(A(\cos\theta)^2)$$

$$\frac{\partial}{\partial y}(A(\sin\theta)^2) = \frac{\partial}{\partial x}(A(\cos\theta)(\sin\theta))$$

which may be expanded as

$$(A(\sin\theta))_y \cos\theta + A(\sin\theta)(\cos\theta)_y = (A\cos\theta)_x \cos\theta + A(\cos\theta)(\cos\theta)_x \tag{9}$$

$$(A(\sin\theta))_y \sin\theta + A(\sin\theta)(\sin\theta)_y = (A\cos\theta)_x \sin\theta + A(\cos\theta)(\sin\theta)_x \tag{10}$$

Multiplying (9) by $\sin\theta$ and (10) by $\cos\theta$ then subtracting yields

$$\begin{aligned} 0 &= A\big[(\sin\theta)^2(\cos\theta)_y - (\sin\theta)(\cos\theta)(\sin\theta)_y - (\sin\theta)(\cos\theta)(\cos\theta)_x \\ &\quad + (\cos\theta)^2(\sin\theta)_x\big] \\ &= A\left[-(\sin\theta)\theta_y + (\cos\theta)\theta_x\right] \end{aligned}$$

The real-analyticity of the functions involved implies two cases. If $A \equiv 0$, this gives

$$u = x + c_1$$

$$v = y + c_2$$

where c_1, c_2 are constants. This is clearly a diffeomorphism. The other case requires

$$-(\sin\theta)\theta_y + (\cos\theta)\theta_x = 0$$

for all $(x, y) \in \mathbb{R}^2$. By Theorem 2.1, we have $\theta = \theta(x, y)$ is a constant function. By making the change of variables

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and $\bar{A}(\bar{x}, \bar{y}) = A(x, y)$, we obtain

$$\begin{bmatrix} \bar{u}_{\bar{x}} & \bar{u}_{\bar{y}} \\ \bar{v}_{\bar{x}} & \bar{v}_{\bar{y}} \end{bmatrix} = \begin{bmatrix} 1 & \bar{A} \\ 0 & 1 \end{bmatrix}.$$

This implies

$$\bar{v} = \bar{y} + c$$

$$\bar{u} = \bar{x} + h(\bar{y})$$

where c is a constant and $h'(\bar{y}) = \bar{A}$. This may be inverted to give

$$\bar{x} = \bar{u} - h(\bar{v} - c)$$

$$\bar{y} = \bar{v} - c.$$

Changing back to the original variables gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u - c\sin\theta - (\cos\theta)h((\sin\theta)u + (\cos\theta)v - c) \\ v - c\cos\theta + (\sin\theta)h((\sin\theta)u + (\cos\theta)v - c) \end{pmatrix}$$

Suppose the $\mathbf{f}$ is a polynomial, namely, u and v are polynomials. Then since θ is constant, equation (8) implies A is a polynomial, and hence so is $\bar{A}$ and h. This implies x and y are polynomial functions of u and v. □

Remark 3.1 It should be noted that the injectivity implied by Theorem 1.1 follows immediately from results related to the Markus-Yamabe Conjecture in dimension two. Olech[12] proved that the Markus-Yamabe Conjecture is equivalent (for $n = 2$) to the statement: If $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and the eigenvalues λ of $\mathbf{f}'(\mathbf{x})$ have $\Re\lambda < 0$ for all λ at all points $\mathbf{x}$, must $\mathbf{f}$ be injective? This result was proved in [5], [8], and [9]. Applying the result to $-\mathbf{f}$ proves the injectivity, and Reduction 1 implies $\mathbf{f}$ is also surjective. It should be noted that the three proofs cited use significantly more advanced techniques than those used in this paper.

Remark 3.2 Theorem 1.1 cannot be extended to maps which are not everywhere defined. Consider the example (see [4])

$$\mathbf{F}(x, y) = \begin{pmatrix} x - \frac{x^2}{2y^2} \\ y - \frac{x}{y} \end{pmatrix}.$$

This function is not defined on the line $y = 0$. The Jacobian $\mathbf{F}'(x, y)$ has both eigenvalues equal to one on its domain. However, $\mathbf{F}(6, 3) = \mathbf{F}(12, -3) = (4, 1)$, so the map is not injective.

Acknowledgement: I would like to thank Gary Hosler Meisters for interesting discussions about this paper and to the Grant Board at Grinnell College for making such discussions possible.

References

[1] H. Bass, E.H. Connell, and D. Wright. The Jacobian conjecture: Reduction of degree and formal expansion of the inverse. *Bull. American Math. Soc.*, 7(2):287–330, 1982.

[2] A. Bialynicki-Birula and M. Rosenlicht. Injective morphisms of real algebraic varieties. *Proc. A.M.S.*, 13:200–203, 1962.

[3] M. Chamberland. Global asymptotic stability, additive neural networks, and the Jacobian conjecture. *Canadian Applied Mathematical Quarterly*, 5:331–339, 1998.

[4] M. Chamberland and G.H. Meisters. A mountain pass to the Jacobian conjecture. *Canadian Mathematical Bulletin*. to appear.

[5] R. Feßler. A proof of the two-dimensional Markus-Yamabe stability conjecture and a generalization. *Ann. Polon. Math.*, 62(1):45–74, 1995.

[6] P.R. Garabedian. *Partial Differential Equations*. Wiley, New York, 1964.

[7] R.A. Garcia, A. Gasull, and A. Guillamon. Geometrical conditions for the stability of orbits in planar systems. *Math. Proc. Camb. Phil. Soc.*, 120:499–519, 1996.

[8] A.A. Glutsyuk. A complete solution of the Jacobian problem for vector fields on the plane. *Russian Mathematical Surveys*, 49(3):185–186, 1994.

[9] C. Gutierrez. A solution of the bidimensional global asymptotic stability conjecture. *Ann. Inst. Henri Poincaré*, 12(6):627–671, 1995.

[10] G.H. Meisters. Jacobian problems in differential equations and algebraic geometry. *Rocky Mountain Journal of Mathematics*, 12:679–705, 1982.

[11] G.H. Meisters. Polymapbib. *World Wide Web, http://www.math.unl.edu/˜gmeister*, 1997.

[12] C Olech. On the global stability of an autonomous system on the plane. *Cont. to Diff. Eq.*, 1:389–400, 1963.

[13] L. Perko. *Differential Equations and Dynamical Systems.* Springer, New York, 1991.

[14] S.I. Pinchuk. Counterexample to the strong real Jacobian conjecture. *Math. Z.*, 217:1–4, 1994.

[15] W. Rudin. Injective polynomial maps are automorphisms. *American Mathematical Monthly*, 102:540–543, 1995.

[16] G. Strang. *Linear Algebra and its Applications.* Harcourt Brace Jovanovich, Inc., San Diego, CA, third edition, 1988.

[17] A. van den Essen. Conjectures and problems surrounding the Jacobian conjecture. In *Recent Results on the Global Asymptotic Jacobian Conjecture.* Dipartimento di Matematica, Università di Trento,Italia, 1993.

[18] D. Wright. On the Jacobian conjecture. *Illinois Journal of Mathematics*, 25:423–440, 1981.

[19] A.V. Yagzhev. Keller's problem. *Siberian Mathematical Journal*, 21:747–754, 1981.