Convex Domains with Stationary Hot Spots [1]

Marc Chamberland

Department of Mathematics and Statistics

McMaster University, Hamilton, Ontario, CANADA, L8S 4K1

E-mail: marc@daniel.math.mcmaster.ca.

and

David Siegel

Department of Applied Mathematics

University of Waterloo, Waterloo, Ontario, CANADA, N2L 3G1

E-mail: dsiegel@math.uwaterloo.ca

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Abstract. Consider the problem of heat flow in a convex domain in $\mathbb{R}^n$ with Dirichlet boundary condition and constant initial temperature. We show that the solution has a fixed hot spot if the domain is invariant under the action of an essential symmetry group.

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1 Introduction

In an open question posed in SIAM Review, M. Klamkin[7] asks whether convex domains with fixed hot spots have centrosymmetry. To be more precise, let Ω be a bounded convex domain in $\mathbb{R}^n$ and let $u : \Omega \times (0, \infty) \to \mathbb{R}$ be the solution to the problem

$$\frac{\partial u}{\partial t} = \Delta u, \qquad (x,t) \in \Omega \times (0,\infty) \tag{1}$$

$$u(x,0) = 1, \qquad x \in \Omega \tag{2}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \in (0,\infty). \tag{3}$$

To ensure uniqueness of a solution to this system, we require that the solution is bounded (see Walter[14, subsection 28.5–28.7]). This will actually imply the function u takes values in the set $[0,1]$. We are interested in characterizing the domains Ω which have a fixed hot spot, that is, the domains where the set

$$P(t) = \left\{ \xi \in \Omega | u(\xi,t) = \max_{x \in \Omega} u(x,t) \right\}$$

consists of a single unchanging point.

Contributions to this problem have recently been made by Gulliver and Willms[4] and Kawohl[5, 6]. Essential to the discussion is the following well-known result:

Theorem 1.1 *(Brascamp and Lieb[2]) The level curves of the solution to (1)–(3) are convex.*

The regularity of the solution together with Theorem 1.1 implies there is exactly one critical point (maximum) for each time $t > 0$. One writes the solution to (1)–(3) in an eigenfunction expansion as

$$u(x,t) = \sum_{i=1}^{\infty} a_i e^{-\lambda_i t} u_i(x) \tag{4}$$

where (λ_i, u_i) are the eigenvalue-eigenfunction pairs for the Dirichlet Laplacian in Ω, and

$$a_i = \int_\Omega u_i(x)dx, \quad i = 1, 2, \cdots.$$

Since the first eigenvalue is simple (ie. has multiplicity one), the scaled function u_1 dominates the solution u for large t, hence if a stationary hot spot exists, it is simply the unique maximum of u_1. Denote this maximum point as P. Using the inequalities of Payne and Stakgold[11] for convex domains, we have

$$\min\{|x - P| : x \in \partial\Omega\} \geq \frac{\pi}{2\sqrt{\lambda_1}},$$

thus giving a rough idea where P is located. The idea of dominating terms in the eigenvalue expansion can be extended to give the following necessary and sufficient condition for a convex domain Ω to have a stationary hot spot. The necessity of the condition is due to Kawohl[6].

Theorem 1.2 *Suppose Ω is a convex domain. Then Ω has a stationary hot spot at P, the critical point of u_1, if and only if*

$$\sum_{k=m}^{n} a_k \nabla u_k(P) = 0 \tag{5}$$

when $\lambda_{m-1} < \lambda_m = \cdots = \lambda_n < \lambda_{n+1}$ for each possible m and n.

The sufficiency follows easily from Theorem 1.1.

Corollary 1.2.1 *Suppose Ω is a convex domain with simple eigenvalues. Then Ω has a stationary hot spot at P, the critical point of u_1, if and only if*

$$\nabla u_k(P) \int_\Omega u_k(x)dx = 0 \tag{6}$$

for $k = 1, 2, \cdots$.

It is not apparent how Theorem 1.2 may be exploited.

Now we consider finding sufficient conditions in terms of the geometry of the domain. In the original question, Klamkin asked whether domains with a fixed hot spot must be centrosymmetric. Gulliver and Willms state that "a body is *centosymmetric* about a point P if for every A on the boundary, there exists another point A' on the boundary such that P is the midpoint of the line segment AA'." In two dimensions, the equilateral triangle is a counter-example to Klamkin's assertion, while in three-dimensions, we consider the regular tetrahedron. In both [4] and [5], theorems are given which state that if n independent $(n-1)$-dimensional hyperplanes of reflection symmetry exist for the region Ω, then Ω has a stationary hot spot, while in [6] it is shown that centrosymmetry of Ω gives the same conclusion. In section 2 we shall generalize these results by showing that if Ω admits an essentially acting symmetry group then the hot spot is fixed for all time. In sections 3 and 4 we explore this sufficient condition in two and three dimensions, respectively. In the final section we give new conjectures on how to characterize such domains.

2 A Sufficient Condition in Terms of Symmetry

As a preliminary result we will use the following:

Theorem 2.1 *Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation such that $g(\Omega) = \Omega$. Then $g(\bar{x}) = \bar{x}$, where*

$$\bar{x} = \frac{\int_\Omega x\,dx}{\int_\Omega dx}$$

is the centroid of the domain Ω.

Proof: Since g is an affine transformation, we have $g(x) = Ax + b$, where A is an orthogonal $n \times n$ matrix and $b \in \mathbb{R}^n$. We then have

$$\begin{aligned}
\int_\Omega x_i dx &= \int_{g(\Omega)} x_i dx \\
&= \int_\Omega \left(\sum_{j=1}^n a_{ij} x_j + b_i \right) |\det A| dx \\
&= \int_\Omega \left(\sum_{j=1}^n a_{ij} x_j + b_i \right) dx \\
&= \sum_{j=1}^n a_{ij} \int_\Omega x_j dx + \int_\Omega b_i dx,
\end{aligned}$$

thus

$$\bar{x}_i = \frac{\int_\Omega x_i dx}{\int_\Omega dx} = \sum_{j=1}^n a_{ij} \bar{x}_i + b_i,$$

which implies $\bar{x} = g(\bar{x})$. □

Theorem 2.1 implies that the symmetries of Ω form a subgroup $\mathcal{G}$ of the orthogonal group $O(n)$. Without loss of generality, we assume Ω has its centroid at the origin.

We say the subgroup $\mathcal{G}$ of $O(n)$ is *essential* if for any $x \neq 0$ there exists a $g \in \mathcal{G}$ such that $g(x) \neq x$. We now offer the main result of this section.

Theorem 2.2 *If Ω admits an essential symmetry group, then the origin is a stationary hot spot.*

To prove this result, we require one supporting lemma.

Lemma 2.1 *If the C^1 function f is invariant under $\mathcal{G}$, an essentially acting subgroup of $O(n)$, then $\nabla f(0) = 0$.*

Proof: The invariance of f implies $f(g(x)) = f(x)$ for $g \in \mathcal{G}$. Since $\mathcal{G}$ is a subgroup of $O(n)$, we know $g(x) = Ax$ for some orthogonal $n \times n$ matrix A. By the chain rule, $\nabla f(g(x))\, Dg(x) = \nabla f(x)$. Setting $x = 0$ gives $A^T \nabla f(0)^T = \nabla f(0)^T$. If $\nabla f(0)^T \neq 0$, then the group $\mathcal{G}$ is not essential, a contradiction. Therefore $\nabla f(0)^T = 0$. □

The proof of Theorem 2.2 now follows.

Proof: Because of the invariance of the heat equation under $O(n)$ and the uniqueness of the boundary value problem, $u(\cdot, t)$ satisfies the conditions of the lemma for each $t \in (0, \infty)$. This implies the origin is a critical point for all time, and thus a stationary hot spot (by Theorem 1.1). □

Some results for our hot spot problem follow immediately from this theorem. If the domain is centrosymmetric or admits n independent $(n-1)$-dimensional hyperplanes of reflection symmetry, then the corresponding subgroup $\mathcal{G}$ generated from these symmetries is essential, so the centroid is a stationary hot spot for that domain. In the following sections, we shall consider how Theorem 2.2 applies in dimensions two and three.

3 Planar Domains

The question of finding domains which admit a stationary hot spot in two dimensions is straightforward in the context of the last section since there is a simple characterization of the finite subgroups of $O(2)$.

Proposition 3.1 *If $\mathcal{G}$ is a finite subgroup of $O(2)$, then it is isomorphic to either $\mathcal{C}_n$ (the cyclic group of order n) or $\mathcal{D}_n$ (the dihedral group of order $2n$) for some positive integer n.*

The proof may be found in Gilbert[3]. Geometrically, the cyclic groups $\mathcal{C}_n$ correspond to rotations by $2\pi/n$ radians. If $n \geq 2$, these groups are clearly essential. Domains which are centrosymmetric admit a symmetry group containing $\mathcal{C}_2$. The cases of the equilateral triangle and the set Ω considered by Gulliver and Willms where $\Omega = \{(r, \theta) | r \leq 11 + \cos(3\theta)\}$ admit a symmetry group containing $\mathcal{C}_3$, thus they have a stationary hot spot.

The dihedral groups $\mathcal{D}_n$ correspond to a rotation and a reflection, and these groups are essential for all positive integers n. If a symmetry group $\mathcal{G}$ is infinite, then the domain Ω must be the disc; this is clear by a density argument on the circle.

The papers of Kawohl and Gulliver and Willms gave results only for domains with reflection symmetries and centrosymmetry. It should be noted that if a domain admits two independent lines of reflection symmetry, then it also admits a symmetry group $\mathcal{C}_n$ for some $n \geq 2$ since two reflections is equivalent to a rotation (see Ryan[12]).

Kawohl argues that, intuitively, a domain with a stationary hot spot will admit a certain amount of symmetry. Kuttler and Sigillito[8] state that regions with symmetries will often have multiple eigenvalues. In light of these remarks, we offer

Theorem 3.1 *If a convex region Ω admits a symmetry group $\mathcal{C}_n$ for some $n \geq 3$, then $\lambda_2 = \lambda_3$.*

Proof: Alessandrini[1] recently proved that for any convex domain $\Omega \subset \mathbb{R}^2$, the unique nodal line of an eigenfunction corresponding to λ_2 must go the boundary of Ω. Since Ω admits a symmetry group $\mathcal{C}_n$ with $n \geq 3$, the nodal line of an eigenfunction would not be invariant under a rotation of $2\pi/n$ about P, hence λ_2 is not simple. □

We cannot expect a higher multiplicity since Lin[9] showed that for convex domains, the multiplicity of λ_2 is at most two. Also, centrosymmetric domains need not have multiple eigenvalues. Consider a rectangle $0 \leq x \leq a$, $0 \leq y \leq b$, where a^2 and b^2 are incommensurate. It is well-known that the eigenvalues of this rectangle are

$$\lambda_{m,n} = \pi^2 \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right], \quad m, n = 1, 2, \cdots.$$

If $\lambda_{m,n} = \lambda_{\bar{m},\bar{n}}$, then

$$\bar{m}^2 - m^2 = \frac{a^2}{b^2}(n^2 - \bar{n}^2).$$

Since a^2/b^2 is irrational, we must have $n = \bar{n}$ and $m = \bar{m}$, thus the eigenvalues of this rectangle are simple.

Lastly, we point out that domains which admit a symmetry group need not admit either a reflection symmetry or centrosymmetry. Consider the

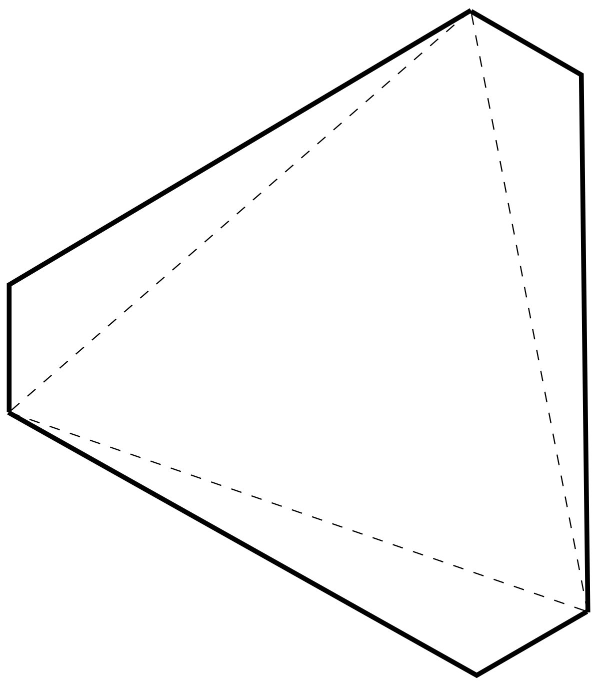

Figure 1: A domain with $\mathcal{C}_3$ symmetry, yet with neither reflection symmetry nor centrosymmetry.

modification of the equilateral triangle in Figure 1; this region admits a $\mathcal{C}_3$ symmetry yet none other.

4 Three Dimensions

The situation for $\Omega \subset \mathbb{R}^3$ is much more challenging than the planar case. We shall cite group-theoretic results from Gilbert[3].

A classical result states that the finite subgroups of $SO(3)$ are isomorphic to the cyclic groups $\mathcal{C}_n$, the dihedral groups $\mathcal{D}_n$, the alternating groups $\mathcal{A}_4$ and $\mathcal{A}_5$, and the symmetric group $\mathcal{S}_4$. The last three groups represent the

Platonic solids. All of these groups, except for C_1, are essential. It is also known that finite subgroups of $O(3)$ contain at least one of these groups, besides C_1.

In three dimensions, however, we cannot conclude that a subgroup of $O(3)$ which is infinite must be a sphere; any volume of revolution admits an infinite symmetry group. It is known that each non-identity element of $SO(3)$ has a unique fixed axis. If the domain Ω admits two different axes of rotation, then Ω admits an essential subgroup of $O(n)$, so it has a stationary hot spot. The same may be said if Ω has an axis of rotation and a plane of reflection symmetry which does not contain the axis.

Lastly, we can modify the regular tetrahedron , as we modified the triangle for the planar case, to obtain a domain which has a stationary hot spot but has no reflection symmetry and is not centrosymmetric. Define a concave function on the equilateral triangle which is rotationally symmetric, yet which admits no reflection symmetry. Also, we require that the gradient of the function is sufficiently small. Superimposing this function on each face of the tetrahedron gives a domain which is convex (the reason for the sufficiently small gradient), and admits the required symmetry to apply the reasoning of the last paragraph, thus yielding a domain with a stationary hot spot. Similar modifications may be made to the other Platonic solids.

5 Conclusion

In light of the results obtained thus far, we make new conjectures:

Conjecture 5.1 If a bounded convex domain $\Omega \subset \mathbb{R}^2$ possesses a stationary hot spot, then there exists $n \geq 2$ such that Ω admits a symmetry group containing $\mathcal{C}_n$.

An affirmation of Conjecture 5.1 would immediately imply the simpler

Conjecture 5.2 If a bounded domain $\Omega \subset \mathbb{R}^2$ has a stationary hot spot and no multiple eigenvalues, then Ω is centrosymmetric.

Conjecture 5.3 If a bounded convex domain $\Omega \subset \mathbb{R}^3$ possesses a stationary hot spot, then either Ω has two independent axes of rotation symmetry or it has one axis of rotation symmetry and one plane of reflection symmetry not containing the axis.

Conjecture 5.4 If a bounded convex domain $\Omega \subset \mathbb{R}^n$ possesses a stationary hot spot, then its symmetry group $\mathcal{G}$ acts essentially.

Conjecture 5.4 implies all the other conjectures.

These conjectures are not the first to suggest that extra conditions on well-posed problems imply certain symmetry on the domain in question. Several theorems or conjectures have been made (see for example [10], [13],[15]) which claim that the existence of a solution to certain over-determined boundary value problems imply the domain is a ball.

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