The Series for e via Integration

This note describes yet another way of obtaining the classical series expansion for e, namely

$$e = \lim_{n\to\infty} \left\{ 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right\}$$

by means of integrals, *not* by Taylor polynomials. This derivation is an interesting application of integration by parts and it offers a nice introduction to infinite series.

Let

$$x_n = \int_0^1 t^n e^{-t} dt$$

for $n = 0, 1, 2, \ldots$ One may easily show that

$$x_0 = -\frac{1}{e} + 1 \tag{1}$$

and integrating by parts yields

$$x_n = -\frac{1}{e} + n x_{n-1}, \quad n \geq 1 \tag{2}$$

Repeated use of this formula gives

$$\begin{aligned} x_n &= -\frac{1}{e} + n\left(-\frac{1}{e} + (n-1)x_{n-2}\right) \\ &= \cdots \\ &= -\frac{1}{e}\left(1 + n + n(n-1) + \cdots + n!\right) + n! x_0 \\ &= -\frac{1}{e}\left(1 + n + n(n-1) + \cdots + n!\right) + n!\left(-\frac{1}{e} + 1\right) \end{aligned}$$

This may be readily verified by induction. By considering the integrand in the definition of x_n, one sees that $|x_n| \leq 1$ for all n. We then have

$$0 = \lim_{n\to\infty} \frac{x_n}{n!} = \lim_{n\to\infty} \left[-\frac{1}{e}\left(\frac{1}{n!} + \frac{1}{(n-1)!} + \cdots + \frac{1}{1!} + 1\right) + 1 \right]$$

and so we obtain

$$e = \lim_{n\to\infty} \left\{ 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \right\}$$