Constructing Equidissections for Certain Classes of Trapezoids

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Abstract

We investigate equidissections of a trapezoid $T(a)$, where the ratio of the lengths of two parallel sides is a. (An *equidissection* is a dissection into triangles of equal areas.) An integer n is in the *spectrum* $S\big(T(a)\big)$ if $T(a)$ admits an equidissection into n triangles. Suppose a is algebraic of degree 2 or 3, with each conjugate over $\mathbf{Q}$ having positive real part. We show that if n is large enough, n is in $S\big(T(a)\big)$ iff $n/(1+a)$ is an algebraic integer. If, in addition, a is the larger root of a monic quadratic polynomial with integer coefficients, we give a complete description of $S\big(T(a)\big)$.

Key words: equidissection, spectrum, principal spectrum

1 Introduction

Suppose $a > 0$ and $T(a)$ is a trapezoid in which there are two parallel sides the ratio of whose lengths is a. Kasimatis and Stein [3] ask: For which a is there a dissection of $T(a)$ into triangles of equal areas? (For fixed a, any two $T(a)$ are affine equivalent, so the question makes sense.) Such a dissection is called an *equidissection* of $T(a)$. The *spectrum* of $T(a)$, denoted $S\Big(T(a)\Big)$, is the set of all integers n so that $T(a)$ has an equidissection into n triangles. Note that if n is in $S\Big(T(a)\Big)$, then so is kn for all $k > 0$. If $S\Big(T(a)\Big)$ consists of precisely the positive multiples of n, we write $S\Big(T(a)\Big) = \langle n \rangle$ and call $S\Big(T(a)\Big)$ *principal.*

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Note also that if m and n are in $S\big(T(a)\big)$, then so is $m+n$. (Divide each of the parallel sides of $T(a)$ in the ratio $m{:}n$. This gives a dissection of $T(a)$ into two trapezoids, each affine equivalent to $T(a)$, with areas in the ratio $m{:}n$.)

The following basic fact about $S\big(T(a)\big)$ is implicit in [3]. Its proof uses standard results on integral closure and valuation rings; a good reference is [5], pp. 64–65 and 71–73.

Theorem 1.1. (Kasimatis and Stein) If n is in $S\big(T(a)\big)$, then $n/(1+a)$ is a positive algebraic integer.

Proof. If K is a field containing $\mathbf{Z}$, the intersection of the valuation rings of K is the integral closure of $\mathbf{Z}$ in K. So it is enough to show that $n/(1+a)$ is contained in each valuation ring V in $\mathbf{R}$. This is the content of the displayed equation on line 2, p. 122 of [8]. We outline the derivation of this equation. Assume the vertices of $T(a)$ are $(0,0)$, $(1,0)$, $(0,1)$, $(a,1)$ and use V, as in [6], to "tricolor" the points of $\mathbf{R}^2$. As in [6], we find that some triangle T in the dissection has vertices of all three colors; consequently, $(1+a)/n = 2$(Area of T) is not in the maximal ideal of V. So $n/(1+a)$ is in V. ∎

Remark 1.1. In particular, if $S\big(T(a)\big)$ is nonempty, then a is algebraic over $\mathbf{Q}$; this is stated explicitly in [3].

Remark 1.2. Note that Theorem 1.1 generalizes the result in [6]: a square cannot be dissected into an odd number of triangles of equal areas.

Suppose a is rational. Say $a = t_1/t_2$ in lowest terms. Theorem 1.1 implies that every n in $S\big(T(a)\big)$ is a multiple of t_1+t_2. But we see immediately that all positive multiples of t_1+t_2 are in $S\big(T(a)\big)$ so $S\big(T(a)\big) = \langle t_1+t_2\rangle$. That is, the converse of Theorem 1.1 holds in this case. (This is Theorem 3.2 of [3].) However, the converse does not hold in general. For it is easy to see that no $n < 1+a$ can lie in $S\big(T(a)\big)$. (Assume the vertices of $T(a)$ are $(0,0)$, $(1,0)$, $(0,1)$, $(a,1)$. Then the area of $T(a)$ is $(1+a)/2$, and an equidissection of $T(a)$ must contain a triangle of area $\le 1/2$, because at least one triangle has a base on the x-axis.) For instance, if $a = 3+2\sqrt{2}$, then $4/(1+a)$ is an algebraic integer but 4 is not in $S\big(T(a)\big)$.

One problem addressed in this paper is:

Find conditions on a which ensure that:
(∗) *Every sufficiently large integer n for which $n/(1+a)$ is an algebraic integer is in $S\big(T(a)\big)$.*

In fact, our first main theorem is as follows:

Theorem I. Statement $(*)$ holds for every a algebraic of degree 2 or 3 over $\mathbf{Q}$ whose conjugates over $\mathbf{Q}$ have positive real parts.

There is a considerable body of literature about polynomials f all of whose roots have positive real parts. (These are the polynomials for which $f(-x)$ is said to be "stable.") It is easy to see that the coefficients of such an f are nonzero and alternate in sign — just factor $f(-x)$ into irreducibles of degree 1 and 2 in $\mathbf{R}[x]$. When $\deg f > 2$, these conditions on the coefficients do not suffice. The additional necessary and sufficient conditions were determined by Routh and Hurwitz; a simpler form of the conditions was given by Liénard and Chipart. (See [1].) But in degree 3 everything is simple; the roots of $t_4x^3 - t_3x^2 + t_2x - t_1$, $t_4 > 0$, all have positive real parts iff each $t_i > 0$ and $t_2t_3 > t_1t_4$. (See Lemma 3.4 below.) It is a remarkable fact, called to our attention by a reviewer, that these polynomials appear in the solution of an apparently unrelated dissection problem. (See [4].)

When a has degree 2, Theorem I is proved using the following result (Theorem 3.3 in [3]).

Theorem 1.2. Suppose that the degree of a is 2 and that the conjugate $\overline{a}$ of a over $\mathbf{Q}$ also is positive. Let $t_3x^2 - t_2x + t_1$ be the irreducible element of $\mathbf{Z}[x]$ with $t_3 > 0$ having a as a root. Then $t_1 + t_2 + t_3$ is in $S\big(T(a)\big)$.

When a has degree 3, our proof of Theorem I depends on the following analogue of Theorem 1.2 which will be established in Section 3.

Theorem 1.3. Suppose that the degree of a is 3 and that each conjugate of a over $\mathbf{Q}$ has positive real part. Let $F = t_4x^3 - t_3x^2 + t_2x - t_1$ be the irreducible element of $\mathbf{Z}[x]$ with $t_4 > 0$ having a as a root. Then all sufficiently large multiples of $t_1 + t_2 + t_3 + t_4$ are in $S\big(T(a)\big)$.

Remark 1.3. Our proof of Theorem 1.3 involves the rational parameterization of a component of an intersection of three quadrics in $\mathbf{P}^5$. The argument does not generalize to a of degree > 3. (For a of degree 4 with all conjugates having positive real parts, there is an analogue, however, involving four quadrics, that has been used by the second author to show that $S\big(T(a)\big)$ is not empty.) It seems doubtful that $(*)$ need hold for a of degree > 4, even assuming that all conjugates of a have positive real parts.

Remark 1.4. There is empirical evidence suggesting that if some conjugate of a over $\mathbf{Q}$ has real part ≤ 0, then $S\big(T(a)\big)$ is empty. Unfortunately, there is not a single a for which we have been able to prove this. (In [8], a weaker conjecture is made: $S\big(T(a)\big)$ is empty if a has a conjugate that is real and negative.)

In [2] and [7], the authors calculate $S\big(T(a)\big)$ for certain a of degree 2. Our second main theorem, proved in Section 2, extends these results by proving a stronger form of Theorem I for a large class of a.

Theorem II. Assume the notation as in Theorem 1.2. If $t_3 = 1$, the following holds: Every n that is greater than both $1 + a$ and $1 + \overline{a}$ for which $n/(1 + a)$ is an algebraic integer is in $S\big(T(a)\big)$.

Remark 1.5. In particular, if $a > \overline{a}$ and $t_3 = 1$, we conclude that n is in $S\big(T(a)\big)$ iff $n/(1 + a)$ is an algebraic integer > 1.

The proofs of Theorem 1.3 and Theorem II have a common element — they involve a preliminary dissection of $T(a)$ into triangles with commensurable areas. Indeed, the same is true of the proof of Theorem 1.2. But where the preliminary dissection related to Theorem 1.2 involves three triangles, the one related to Theorem 1.3 involves six. And the most difficult subcase of Theorem II involves a preliminary 7-triangle dissection.

We conclude this introduction by showing how Theorem I follows from Theorems 1.2 and 1.3. We treat the case where a has degree 3, making use of Theorem 1.3; the argument for a of degree 2 is virtually identical.

Proof. Let a and F be as in Theorem 1.3 and set $T = t_1 + t_2 + t_3 + t_4$. The integers n for which $n/(1 + a)$ is an algebraic integer form a subgroup of $\mathbf{Z}$ containing T; let $m > 0$ be a generator for this subgroup. Now $1/(1 + a)$ is a root of $-x^3F(1/x-1)$, which has the form Tx^3+bx^2+cx+d, where b, c, d, and T have no common factor. Thus $m/(1+a)$ is a root of $Tx^3+bmx^2+cm^2x+dm^3$. Since $m/(1+a)$ is a root of a monic irreducible polynomial in $\mathbf{Z}[x]$ of degree 3, T divides bm, cm^2, and dm^3. So if q is a prime dividing T, then q must divide m. (For otherwise q divides b, c, and d as well as T, and these four integers have no common factor.)

To continue we will use the following elementary fact. Let X be a subset of the positive integers that is closed under addition and contains for each prime p an integer prime to p. Then X contains all large integers. (Choose D in X. The image of X in the cyclic group $C = \mathbf{Z}/D$ is, like X, closed under addition and is therefore a subgroup; this subgroup is cyclic generated by some r dividing D. Since r divides all elements of X, $r = 1$. So the image of X is all of C and X contains an element in each mod D congruence class. But whenever j is in X, so is $j + D$.) Applying this to the set $m^{-1}S\big(T(a)\big)$, we see that it suffices to show that for each p, $S\big(T(a)\big)$ contains some Nm with $(N, p) = 1$. If p does not divide m, then as we have seen p does not divide T. So there are arbitrarily large multiples of Tm prime to p, and Theorem 1.3 gives the desired result. Now assume p divides m.

Take k large and prime to p. Our idea is to find a trapezoid $T(\theta)$ inside $T(a)$, as in the next paragraph, with the area of $T(a)$ equal to $\big(\text{Area of } T(\theta)\big)\ \big(mk/(mk-1)\big)$. To this end, we define θ by $1/(1+\theta) = \big(k/(mk-1)\big)\ \big(m/(1+a)\big)$. Then $mk\theta = (mk-1)a - 1$, and since k is large, the real parts of θ and its conjugates, like those of a and its conjugates, are positive. Also, $1/(1+\theta)$ is a root of the polynomial $G = T(mk-1)^3x^3 + bmk(mk-1)^2x^2 + c(mk)^2(mk-1)x + d(mk)^3$. Since T divides bm, cm^2, and dm^3, $G/T = g$ has integer coefficients. The irreducible h in $\mathbf{Z}[x]$ with $h(\theta) = 0$ is the quotient of $E(x) = (1+x)^3 g\big(1/(1+x)\big)$ by an integer. Since $E(-1) = (mk-1)^3$, the alternating sum of the coefficients of h divides $(mk-1)^3$. Theorem 1.3 applied to θ now shows that $M(mk-1)^3$ is in $S\big(T(\theta)\big)$ for some M prime to p.

Now we may choose the trapezoids $T(a)$ and $T(\theta)$ so that they have three vertices in common, with $T(a)$ being the nonoverlapping union of $T(\theta)$ and a triangle. Then the area of the triangle is $\big(\text{Area of } T(\theta)\big)/(mk-1)$. Thus the dissection of $T(\theta)$ into $M(mk-1)^3$ triangles of equal areas extends to a dissection of $T(a)$ into $M(mk-1)^2km$ triangles of equal areas. But p does not divide k, M, or $mk-1$ (since p divides m), and the result is proved. ∎

2 Proof of Theorem II

Suppose that a has degree 2 and that the conjugate $\overline{a}$ of a over $\mathbf{Q}$ is positive. Let $t_3x^2 - t_2x + t_1$ be the irreducible element of $\mathbf{Z}[x]$ with $t_i > 0$ having a as a root.

Now $(t_1+t_2+t_3)/(1+a)$ is a root of $x^2 - (t_2+2t_3)x + t_3(t_1+t_2+t_3) = 0$. So the subgroup of $\mathbf{Z}$ consisting of those n with $n/(1+a)$ an algebraic integer contains $t_1+t_2+t_3$ and is generated by some $m = (t_1+t_2+t_3)/d$. We describe d explicitly. It evidently is the largest D dividing $t_1+t_2+t_3$ for which $(1/D)(t_1+t_2+t_3)/(1+a)$ is an algebraic integer, i.e., for which $x^2 - (1/D)(t_2+2t_3)x + (1/D^2)\big(t_3(t_1+t_2+t_3)\big)$ has integer coefficients. Now if D divides $t_1+t_2+t_3$ and t_2+2t_3, it must be prime to t_3. So D^2 divides $t_3(t_1+t_2+t_3)$ iff it divides $t_1+t_2+t_3$. We conclude that $n/(1+a)$ is an algebraic integer iff $m = (t_1+t_2+t_3)/d$ divides n, where d is the largest D such that D^2 divides $t_1+t_2+t_3$ and D divides t_2+2t_3.

Write $t_1+t_2+t_3 = d^2r$ and $t_2+2t_3 = ds$. From Theorem 1.1 and from an argument virtually identical to the one given at the end of Section 1, we have:

Theorem 2.1. Every element of $S\big(T(a)\big)$ is a multiple of dr.

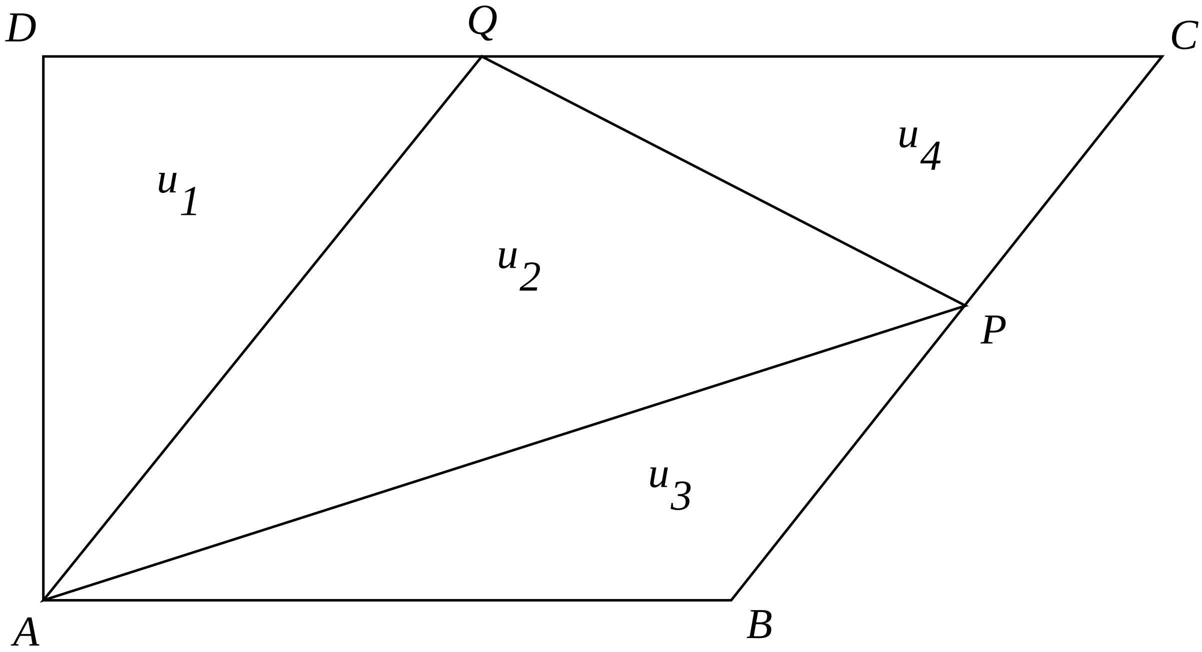

Figure 1.

Theorem 2.2. All sufficiently large multiples of dr are in $S\big(T(a)\big)$.

Remark 2.1. Suppose $d = 1$. This occurs, for example, when $t_1 + t_2 + t_3$ is square-free. Then $S\big(T(a)\big) = \langle t_1 + t_2 + t_3 \rangle$. For instance, if $a^2 - 11a + 3 = 0$, then $S\big(T(a)\big) = \langle 15 \rangle$. (This example appears in [3].)

Remark 2.2. The only way for $S\big(T(a)\big)$ to be principal is to have dr lie in $S\big(T(a)\big)$. That is, $S\big(T(a)\big)$ is principal iff $S\big(T(a)\big) = \langle dr \rangle$.

For the rest of this section we assume that $t_3 = 1$. Let $b = 1 + a$, $\overline{b} = 1 + \overline{a}$. Then $b + \overline{b} = t_2 + 2t_3 = ds$ and $b\overline{b} = t_1 + t_2 + t_3 = d^2 r$. Hence b and $\overline{b}$ are the roots of $x^2 - dsx + d^2 r = 0$.

Theorem 2.3. Suppose $1 \le k \le d$. If b and $\overline{b}$ are $\le kdr$, then kdr is in $S\big(T(a)\big)$.

Proof: Let U be the trapezoid with vertices $A = (0, 0)$, $B = (kdr/b, 0)$, $C = (kdr(b-1)/b, 2)$, $D = (0, 2)$. The area of U is kdr and the ratio of the bases is $b - 1 = a$. Thus it suffices to dissect U into triangles of area 1. To do this, we construct points P and Q so that the areas u_1, u_2, u_3, u_4 of the four triangles in Figure 1 are integers.

As P moves along the right edge from B to C, u_3 increases from 0 to $kdr/b \ge 1$. We may choose P so that $u_3 = 1$. As Q moves along the top edge from D to C,

u_1 increases from 0 to $kdr - (kdr)/b = kdr - (k\overline{b})/d \geq kr(d-k) \geq 0$. Choose Q so that $u_1 = kr(d-k)$. Now it is enough to show that u_4 is an integer. The height of the triangle in question is $2 - (2b)/(kdr)$, while its base is $kdr(b-1)/b - kr(d-k) = kdr(k/d - 1/b)$. Thus $u_4 = (kdr - b)(k/d - 1/b) = k^2r - k(dr/b + b/d) + 1$. Now $b^2 - dsb + d^2r = 0$ so $s = dr/b + b/d$. Thus $u_4 = k^2r - ks + 1$, completing the proof. ∎

Remark 2.3. In fact, the restriction $k \leq d$ in Theorem 2.3 is unnecessary. This is the content of Theorem 2.7 below, the main result of this section.

Corollary. If b and $\overline{b}$ are $\leq dr$, then $S\big(T(a)\big)$ is principal.

Proof. Taking $k = 1$ in Theorem 2.3, we have that dr is in $S\big(T(a)\big)$ The conclusion now follows from Theorem 2.1. ∎

Remark 2.4. Conversely, suppose $b > dr$. Then dr cannot lie in $S\big(T(a)\big)$. (We observed this in the paragraph after Remark 1.2.) It follows from Remark 2.2 that $S\big(T(a)\big)$ is not principal. We do not know whether the same conclusion holds when $\overline{b} > dr$. Indeed, whether $T(a)$ and $T(\overline{a})$ always have the same spectrum is an open problem.

Remark 2.5. Theorem 2.7 will show that kdr is in $S\big(T(a)\big)$ whenever $kdr \geq$ both b and $\overline{b}$. When $kdr < b$, the argument noted in Remark 2.4 shows that kdr is not in $S\big(T(a)\big)$. Using Theorem 2.1 we conclude: If $a > \overline{a}$, then $S\big(T(a)\big)$ consists of all multiples kdr of dr that are greater than $1+a$. But whether any multiples of dr that are less than $1+a$ can lie in $S\big(T(\overline{a})\big)$ is an open problem.

Before continuing, we introduce some notation. An *oriented triangle* is an ordered triple of points $\big((x_1,y_1),\ (x_2,y_2),\ (x_3,y_3)\big)$. The *area* of the oriented triangle is the (possibly negative) determinant $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

The following somewhat technical theorem paves the way for the more general results that follow.

Theorem 2.4. Suppose $k \geq d$. If $ds - k - 1 - r(k-d)^2 \geq 0$, then kdr is in $S\big(T(a)\big)$.

Proof. Let U be the same trapezoid as in Theorem 2.3. For points P and Q in the plane, let $u_1, \ldots, u_6$ be the areas of the oriented triangles (A,Q,D), (A,P,Q), (A,B,P), (C,P,B), (C,Q,P), and (C,D,Q). Figure 2 illustrates the situation when each $u_i \geq 0$.

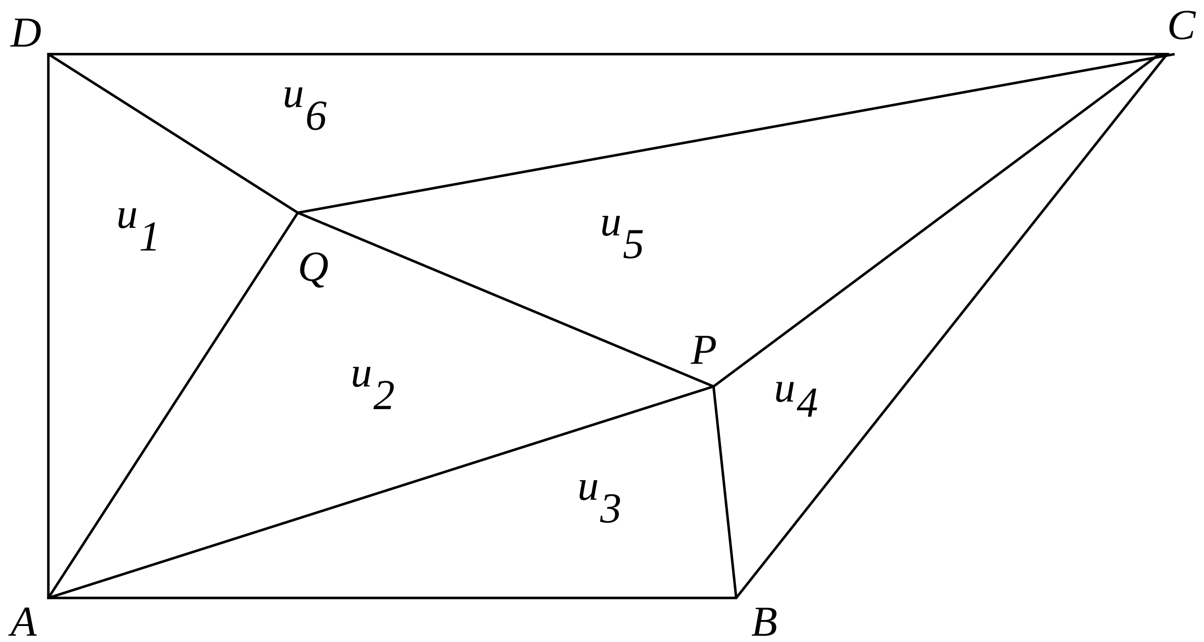

Figure 2.

It suffices to choose $P = (x, y)$ and $Q = (u, v)$ so that the u_i are nonnegative integers. With u an integer to be determined later, choose v so that $u_6 = t_1$. Next choose x and y so that $u_3 = 1$ and $u_4 = k - 1$.

We calculate $u_2 = (xv - yu)/2$. To do this, we express y, x, and v as linear combinations of 1 and b with rational coefficients. Note first that $u_3 = 1$ gives $y = \big(2/(kdr)\big)b$. We claim that:

$$x = (ks - k - 1) - \left(\frac{k-d}{d}\right)b, \tag{1}$$

$$\frac{1}{2}v = \left(\frac{k-d}{k}\right) + \left(\frac{1}{kdr}\right)b. \tag{2}$$

To prove (1), note that $2k - 2 = 2u_4 = \begin{vmatrix} \frac{kdr(b-1)}{b} & 2 & 1 \\ x & \frac{2b}{kdr} & 1 \\ \frac{kdr}{b} & 0 & 1 \end{vmatrix}$. Solve for x to get $x = b + kdr/b - k - 1$. Since $s = dr/b + b/d$, we get (1). To prove (2), note that $u_6 = t_1$ says $v/2 = 1 - \big(1/(kdr)\big)\big(bt_1/(b-1)\big)$. Now $t_1 = d^2r - ds + 1 = dsb - b^2 - ds + 1 = (b-1)(ds - b - 1)$, so $v/2 = 1 - \big(1/(kdr)\big)\big(b(ds - b - 1)\big)$. Since $bds = d^2r + b^2$, $b(ds - b - 1) = d^2r - b$. Hence $v/2 = 1 - (1/kdr)(d^2r - b)$, which gives (2).

Multiplying (1) and (2) and again using $b^2 = dsb - d^2r$, we find that the coefficient of b in $xv/2$ is $\big(1/(kdr)\big)\big(ds-k-1-r(k-d)^2\big)$. Also, the coefficient of b in $yu/2$ is $u/(kdr)$. Thus, if we choose $u = ds - k - 1 - r(k-d)^2$, then $(xv - yu)/2$ is rational. Further, by the hypothesis, $u_1 = u$ is a nonnegative integer.

Now the coefficient of 1 in $xv/2$ is $\big((k-d)/(kdr)\big)\big(dr(ks-k-1)+dr\big) = (k-d)(s-1)$, while the coefficient of 1 in $yu/2$ is 0. This says that $u_2 = (k-d)(s-1)$, a nonnegative integer.

Since the sum of the u_i is kdr, we know that u_5 is an integer. It remains to show that u_5 is nonnegative. Now $u_1+u_6 = \big(ds-k-1-r(k-d)^2\big)+(d^2r-ds+1) = 2kdr - k^2r - k$, and $u_3+u_4 = k$. So $u_2+u_5 = kdr - (2kdr - k^2r) = (k-d)kr$. Thus $u_5 = (k-d)(kr - s + 1)$. Since $t_1 = d^2r - ds + 1 \geq 1$, $dr \geq s$, and so $kr \geq dr \geq s$. Hence u_5 is nonnegative, and the proof is complete. ∎

Set $k_0 = \lceil \max\{b, \bar{b}\}/(dr) \rceil$. Since both b and $\bar{b}$ are greater than 1 and $b\bar{b} = d^2r$, both b and $\bar{b}$ are less than d^2r. Thus $k_0 \leq d$. Theorem 2.3 says that kdr is in $S\big(T(a)\big)$ for $k_0 \leq k \leq d$. We now show:

Theorem 2.5. Suppose $d+1 \leq k \leq d+k_0-1$. Then kdr is in $S\big(T(a)\big)$, except perhaps when b is a root of $x^2 - d^2x + d^2 = 0$ and $k = 2d-1$.

Proof: We know that either $b/(dr)$ or $\bar{b}/(dr)$ is $> k_0 - 1$. Then $s = (b + \bar{b})/d > r(k_0-1)$. Since $k_0 - 1 \geq k - d$, $s \geq r(k-d)+1$. It follows that the integer $u = ds - k - 1 - r(k-d)^2$ in the statement of Theorem 2.4 is $\geq d\big(r(k-d)+1\big) - k - 1 - r(k-d)^2 = r(k-d)(2d-k) + d - k - 1$. Since $d+1 \leq k \leq 2d-1$, $r(k-d)(2d-k) \geq r(d-1)$. Thus $u \geq r(d-1)+d-k-1$.

Assume first that $k \neq 2d-1$. Then $k \leq 2d-2$, so that $u \geq r(d-1) + d - (2d-2) - 1 = (r-1)(d-1) \geq 0$, and we apply Theorem 2.4. Now suppose $k = 2d-1$. Then $u \geq r(d-1) - d$. Since $d+1 \leq k \leq 2d-1$, $d \geq 2$. If $r \geq 2$, then $u \geq 0$, and again we apply Theorem 2.4. Suppose finally that $r = 1$. By the preceding paragraph, $s \geq r(k-d)+1 = d$. On the other hand, $t_1 = d^2r - ds + 1 \geq 1$ says $d \geq s$. Hence $s = d$, and so $b^2 - d^2b + d^2 = 0$. ∎

Theorem 2.6. In the exceptional case of Theorem 2.5, kdr is in fact in $S\big(T(a)\big)$.

Proof. Let U be the trapezoid in Theorem 2.3 with $k = 2d-1$ and $r = 1$. The idea is to dissect U into seven triangles of integer areas according to the scheme shown in Figure 3. First choose $P = (x, y)$ so that the oriented triangles (A, B, P) and (C, P, B) have areas 1 and $2d-4$. The oriented triangle (A, B, C) has irrational area $(2d-1)d/b$. If A, P, and C were collinear, this area would be $2d-3$, a contradiction. The noncollinearity allows us to find

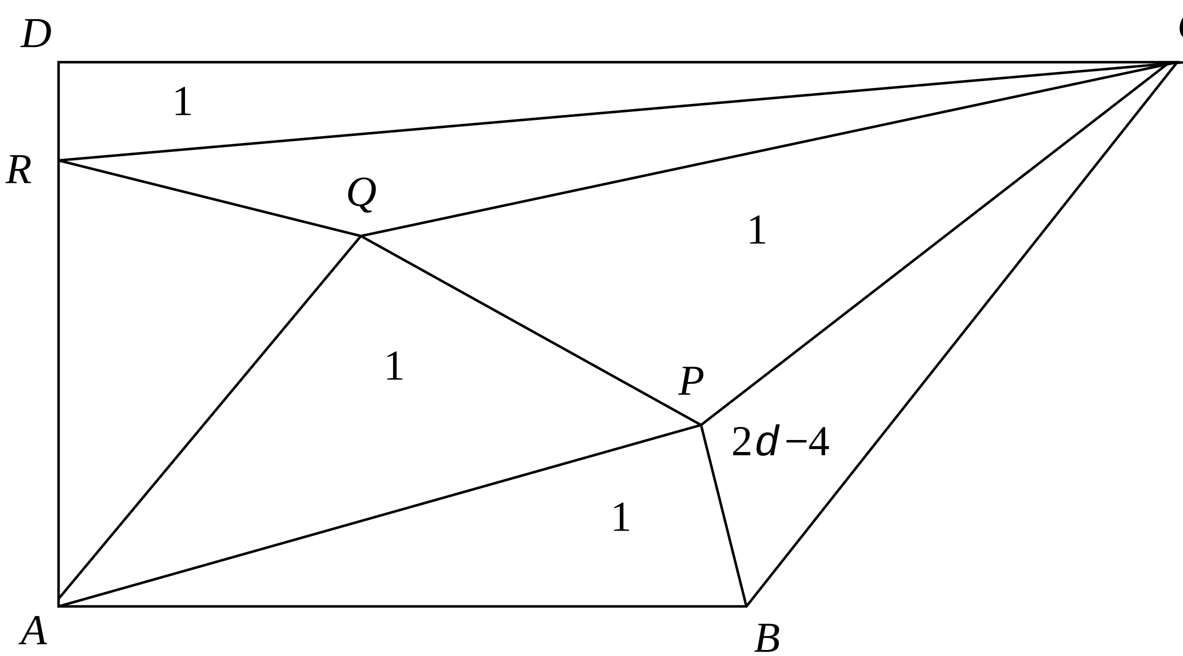

Figure 3.

$Q = (u, v)$ so that (A, P, Q) and (C, Q, P) each have area 1. We further choose $R = (0, w)$ so that (C, D, R) has area 1.

We calculate the area of the oriented triangle (A, Q, R). This area is $wu/2$, and we calculate this by writing w and u as rational linear combinations of 1 and b. Because the procedure is similar to that in Theorem 2.4, we omit some details. At certain steps, we use $b^2 = d^2(b-1)$ and $1/b = 1 - (1/d^2)b$.

To simplify notation, let $\alpha = d(2d-1)/b$, $\beta = d(2d-1)(b-1)/b$. Note first that $1 = \frac{1}{2}(2-w)\beta$ gives $w/2 = 1 - 1/\beta$ so

$$\frac{1}{2}w = \left(\frac{1}{2d-1}\right)\left(\left(d-1\right)+\left(\frac{1}{d}\right)b\right). \tag{3}$$

Next, $1 = \frac{1}{2}\alpha y$ says $\alpha y = 2$ and $\beta y = 2(b-1)$. Now note that $2(2d-4) = \begin{vmatrix} \beta & 2 & 1 \\ x & y & 1 \\ \alpha & 0 & 1 \end{vmatrix}$. Solve for x to get $x = (2d^2 - 3d + 2) - \big((d-1)/d\big)b$.

Now $2 = \begin{vmatrix} 0 & 0 & 1 \\ x & y & 1 \\ u & v & 1 \end{vmatrix}$ gives

$$xv - yu = 2, \tag{4}$$

while $2 = \begin{vmatrix} \beta & 2 & 1 \\ u & v & 1 \\ x & y & 1 \end{vmatrix}$ after simplification gives

$$\beta v - 2u = 2(b + 1 - x). \tag{5}$$

Solve (4) and (5) to get $u = x + (2x - \beta)/(b - 1 - x)$. Let $h = 2d^2 - 3d + 2$ and note that $h - 1 = (2d - 1)(d - 1)$. Then $x = h - \big((d-1)/d\big)b$, and we compute $(2x - \beta)/(b - 1 - x) = (-2 + 1/h) + \big((d-1)/(dh)\big)b$. Thus

$$u = \frac{(2d-1)(d-1)^2}{h}\left(\Big(2d - 1\Big) - \left(\frac{1}{d}\right)b\right). \tag{6}$$

Multiplying (3) and (6), we get

$$\begin{aligned} \frac{1}{2}wu &= \frac{(d-1)^2}{h}\Big((d-1)(2d-1) + \frac{1}{d}\Big((2d-1) - (d-1)\Big)b - (b-1)\Big) \\ &= \frac{(d-1)^2}{h}(h - 1 + 1) = (d-1)^2. \end{aligned}$$

Since the sum of the areas of the seven oriented triangles in Figure 3 is $kdr = 2d^2 - d$, (C, R, Q) has area $d^2 - d - 1$, and we are done. ∎

Theorem 2.7. The spectrum $S\big(T(a)\big)$ contains all kdr with $k \geq k_0$.

Proof. By Theorems 2.3, 2.5, and 2.6, kdr is in $S\big(T(a)\big)$ for $k_0 \leq k \leq k_0 + d - 1$. For larger k, write $k = (k - d) + d$ and use induction on k. ∎

Theorem II follows directly from Theorem 2.7; every n for which $n/(1 + a)$ is an algebraic integer > 1 is a multiple of $m = dr$.

Remark 2.6. As an example, consider the case where $t_1 = 1$. Let a be the larger root of $x^2 - t_2 x + 1 = 0$, $t_2 \geq 3$. Write $t_2 + 2 = d^2 r$, where r is square-free. Then $b = 1 + a > t_2 = d^2r - 2 > (d-1)dr$. So $k_0 = d$, and $S\big(T(a)\big)$ consists of all multiples of $dr \geq d^2r$. For instance, if $a^2 - 7a + 1 = 0$, then $S\big(T(a)\big) = \{9, 12, 15, 18, \ldots\}$. (To show that 15 is in the spectrum, we need the exceptional case in Theorem 2.6.)

Remark 2.7. When $a > \overline{a}$ and $t_3 > 1$, no precise description of $S\big(T(a)\big)$ is known to us. For example, when $7a^2 - 22a + 7 = 0$, we do not know if 6 lies in $S\big(T(a)\big)$. When $8a^2 - 33a + 8 = 0$ or $9a^2 - 31a + 9 = 0$, we do not know if 7 lies in $S\big(T(a)\big)$.

3 Proof of Theorem I

Let s_1, s_2, s_3, s_4 be polynomials in u_1, u_2, u_3, u_4, u_5, u_6 defined as follows, where $t = u_1 + u_2 + u_3 + u_4 + u_5 + u_6$:

$$\begin{aligned}
s_1 &= u_6(t - u_3 - u_4),\\
s_2 &= u_3u_6 + t(u_1 + u_5) + 2u_4u_6 - u_1u_3,\\
s_3 &= u_3u_6 + t(u_2 + u_4) - u_4u_6 + 2u_1u_3,\\
s_4 &= u_3(t - u_1 - u_6).
\end{aligned}$$

The following result helped us discover Theorem 2.4 of the previous section.

Lemma 3.1. Suppose $u_1, \ldots, u_6$ are positive and a is a real root of the polynomial $s_4x^3 - s_3x^2 + s_2x - s_1$. Let $T(a)$ be a trapezoid of area t with parallel sides the ratio of whose lengths is a. Then $T(a)$ can be dissected into triangles of areas $u_1, \ldots, u_6$. In particular, if the u_i are integers, then t lies in $S\big(T(a)\big)$.

Proof. Let U be the trapezoid in Figure 2 with $A = (0, 0)$, $B = (t/(1+a), 0)$, $C = (ta/(1 + a), 2)$, $D = (0, 2)$. We take $P = (x, y)$ so that the oriented triangles (A, B, P) and (C, P, B) have areas u_3 and u_4. Take $Q = (u, v)$ so that the oriented triangles (A, Q, D) and (C, D, Q) have areas u_1 and u_6. If $b = 1 + a$, then u, y, and v are u_1, $2u_3b/t$, and $2 - 2u_6b/(ta)$. Furthermore,

$$2u_4 = \begin{vmatrix} ta/b & 2 & 1 \\ x & y & 1 \\ t/b & 0 & 1 \end{vmatrix}, \text{ and so } x = t/b + au_3 - u_3 - u_4.$$

We claim that it suffices to show that the area of (A, P, Q) is u_2. For the sum of the areas of the six oriented triangles is the area $t = u_1+u_2+u_3+u_4+u_5+u_6$ of the trapezoid, and it would then follow that the area of (C, Q, P) is u_5. Since the u_i are positive, it would further follow (say by a winding number argument) that the picture is as shown in Figure 2 and that we have a dissection.

We set $I = \text{area } (A, P, Q) - u_2 = (xv - yu)/2 - u_2$ and show $I = 0$. Now $abtI = (bx)(atv/2) - ab^2u_1u_3 - abtu_2 = (t + abu_3 - bu_3 - bu_4)(at - bu_6) - ab^2u_1u_3 - abtu_2$. Replace b by $1+a$ and expand each term in powers of a to get $\Big(u_3a^2 - u_4a + (t - u_3 - u_4)\Big)\Big((t - u_6)a - u_6\Big) - u_1u_3(a^3 + 2a^2 + a) - tu_2(a^2 + a)$. The coefficients of a^3, 1, and a^2 in this last polynomial are evidently $s_4, -s_1$, and $-s_3$. And the coefficient of a is $(t - u_3 - u_4)(t - u_6) + u_4u_6 - u_1u_3 - tu_2$. Now $(t - u_3 - u_4)(t - u_6) - tu_2 = t(t - u_2 - u_3 - u_4 - u_6) + u_3u_6 + u_4u_6 = t(u_1 + u_5) + u_3u_6 + u_4u_6$. So the coefficient of a is $t(u_1 + u_5) + u_3u_6 + 2u_4u_6 - u_1u_3 = s_2$. Thus $abtI = s_4a^3 - s_3a^2 + s_2a - s_1$, which is 0 by hypothesis. ∎

Remark 3.1. If we assume more generally that the u_i are nonnegative and not all 0, then the same argument shows that $T(a)$ admits a dissection into triangles whose areas are the nonzero u_i.

Lemma 3.2. Let t_1, t_2, t_3, t_4 be real numbers with $T = t_1 + t_2 + t_3 + t_4 \neq 0$. Let x and y be indeterminates over $\mathbf{R}$. Then the five equations

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = T, \tag{7}$$
$$s_i\,(z_1, \ldots, z_6) = t_iT,\ \ i = 1, 2, 3, 4, \tag{8}$$

have a unique solution with $z_3 = Tx$, $z_4 = Ty$, and each z_i in $\mathbf{R}(x, y)$. Further, if we define $v_i = x(1 - x - y)z_i$, $i = 1, \ldots, 6$, then each v_i is in $\mathbf{R}[x, y]$ and has degree ≤ 3. If the t_i are in $\mathbf{Z}$, the v_i are in $\mathbf{Z}[x, y]$.

Proof. The sum of the four equations in (8) is $(z_1 + \cdots + z_6)^2 = T^2$. As this follows from (7), we may omit the equation $s_2 = t_2T$ without changing the solution set. We then seek z_1, z_2, z_5, z_6 satisfying

$$z_6\ \ (T - Tx - Ty) = t_1T, \tag{9}$$
$$Tx\ (T - z_1 - z_6) = t_4T, \tag{10}$$
$$Tz_2 + T^2y + (Tx - Ty)z_6 + 2Txz_1 = t_3T, \tag{11}$$
$$z_1\ + z_2 + Tx + Ty + z_5 + z_6 = T. \tag{12}$$

From (9), we get $z_6 = t_1/(1 - x - y)$, and (10) tells us that $z_1 = T - z_6 - t_4/x$. Also, (11) and (12) give $z_2 = t_3 - Ty - (x - y)z_6 - 2xz_1$ and $z_5 = T(1 - x - y) - z_1 - z_2 - z_6$. The result follows easily. ∎

Remark 3.2. The v_i defined in Lemma 3.2 satisfy:

(a) $v_1 + v_2 + v_3 + v_4 + v_5 + v_6 = Tx(1 - x - y)$,

(b) $s_i(v_1,\ldots,v_6) = x^2(1-x-y)^2 s_i(z_1,\ldots,z_6) = Tx^2(1-x-y)^2 t_i$,
(c)

$$\begin{aligned}
v_3 &= Tx^2(1-x-y),\ v_4 = Txy(1-x-y),\ v_6 = t_1 x,\\
v_1 &= Tx(1-x-y) - t_1 x - t_4(1-x-y),\\
v_2 &= t_3 x(1-x-y) - Txy(1-x-y) - (x-y)t_1 x - 2xv_1,\\
v_5 &= Tx(1-x-y)^2 - v_1 - v_2 - v_6.
\end{aligned}$$

Lemma 3.3. Let the situation be as in Lemma 3.2 with $t_i > 0$ and $t_2t_3 > t_1t_4$. Then there is a point x in the open interval $(0,1)$ for which each of the $v_i(x,0)$, $i = 1,2,3,5,6$, is positive.

Proof. By Remark 3.2(c), we need only concern ourselves with v_1, v_2, and v_5. Also, $v_1(x,0) = Tx(1-x) - t_1x - t_4(1-x) = -Tx^2 + (t_2+t_3+2t_4)x - t_4$, while $v_2(x,0) = t_3x(1-x) - t_1x^2 - 2xf$, where $f = v_1(x,0)$. So $v_2(x,0) = x(-2f-(t_1+t_3)x+t_3)$. An easy calculation now shows that $v_5(x,0) = -Tx^3 + (t_2+2t_3+3t_4)x^2 - (t_3+3t_4)x + t_4$. So we need only find a point of $(0,1)$ at which f, g, and h are all positive, where:

$$\begin{aligned}
f &= -Tx^2 + (t_2+t_3+2t_4)x - t_4,\\
g &= -2f - (t_1+t_3)x + t_3,\\
h &= -Tx^3 + (t_2+2t_3+3t_4)x^2 - (t_3+3t_4)x + t_4.
\end{aligned}$$

Note that $(t_1+t_3)^2 f\big(t_3/(t_1+t_3)\big) = t_1(t_2t_3-t_1t_4)$ and that $(t_3+t_4)^2 f\big(t_4/(t_3+t_4)\big) = t_4(t_2t_3-t_1t_4)$. Hence f is positive at $t_3/(t_1+t_3)$ and $t_4/(t_3+t_4)$. Since $f(0)$ and $f(1)$ are negative, f has roots c_1 and c_2 in $(0,1)$ with $c_1 < t_3/(t_1+t_3) < c_2$; furthermore, $c_1 < t_4/(t_3+t_4) < c_2$.

Now $g + 2f = t_3 - (t_1+t_3)x$. Setting $x = c_1$, we find that $g(c_1) > 0$. Similarly, using the identity $h - xf = (1-x)(t_4 - (t_3+t_4)x)$, we find that $h(c_1) > 0$. Choose an x in the interval (c_1,c_2) close to c_1. Since x is in (c_1,c_2), $f(x) > 0$. Since x is close to c_1, $g(x)$ and $h(x)$ are > 0. ∎

Lemma 3.4. Let $a = a_1$ be an algebraic number of degree 3 such that a and its conjugates a_2 and a_3 all have positive real parts. Let $F = t_4x^3 - t_3x^2 + t_2x - t_1$ be the irreducible element of $\mathbf{Z}[x]$ with $t_4 > 0$ having a_1, a_2, a_3 as roots. Then each $t_i > 0$ and $t_2t_3 > t_1t_4$.

Proof. F/t_4 factors as $(x-\alpha)(x^2-\beta x+\gamma)$ with α, β, γ positive. This is $x^3 - (\alpha+\beta)x^2 + (\alpha\beta+\gamma)x - \alpha\gamma$, and we note that $\alpha+\beta$, $\alpha\beta+\gamma$, $\alpha\gamma$, and $(\alpha+\beta)(\alpha\beta+\gamma) - \alpha\gamma = \beta(\alpha^2+\alpha\beta+\gamma)$ are all positive. ∎

Lemma 3.5. Let a, F, and $t_1,\ldots,t_4$ be as in Lemma 3.4. Then there are positive integers $u_1,\ldots,u_6$ such that if $s_j = s_j(u_1,\ldots,u_6)$, $j = 1,2,3,4$, then

$s_4a^3 - s_3a^2 + s_2a - s_1 = 0$.

Proof. Let $v_1, \ldots, v_6$ be the polynomials of Lemma 3.2 for our t_1, t_2, t_3, and t_4. The set V of all points in $(0,1)\times(0,1)$ at which each v_i is positive is open. Lemmas 3.4 and 3.3 gives us an x in $(0,1)$ for which $v_1(x,0)$, $v_2(x,0)$, $v_3(x,0)$, $v_5(x,0)$, and $v_6(x,0)$ are all positive. Since $v_4(x,y) = Txy(1-x-y)$, each $v_i(x,y)$ is positive when y is small and positive. So V is nonempty, and we may choose a point (x,y) in V with x and y rational. Let D be a common denominator for x and y, and let $u_i = D^3v_i(x,y)$. Since v_i are elements of $\mathbf{Z}[x,y]$ of degree ≤ 3, the u_i are integers. Furthermore, $s_i(u_1,\ldots,u_6) = D^6s_i(v_1,\ldots,v_6)$; by remark 3.2(b), this is $D^6Tx^2(1-x-y)^2t_i$. Since $t_4a^3 - t_3a^2 + t_2a - t_1 = 0$, we have $s_4a^3 - s_3a^2 + s_2a - s_1 = 0$. ∎

Lemma 3.6. Let p be a prime. Then the u_i of Lemma 3.5 can be chosen so that their sum t is NT, with N an integer prime to p.

Proof. Let D be a large integer prime to p. Because V is open and nonempty and D is large, we can choose (x,y) in V so that Dx and $-Dy$ are integers $\equiv 1 \pmod{p}$. Once again we set $u_i = D^3v_i(x,y)$. Then $t = D^3(v_1+\cdots+v_6) = D^3Tx(1-x-y)$. Hence $t/T = D(Dx)(D-Dx-Dy)$, a product of three integers each prime to p. ∎

The proof of Theorem 1.3 is now direct. Let a, t_1, t_2, t_3, t_4 be as in the statement of Theorem 1.3 and let $T = t_1+t_2+t_3+t_4$. Lemmas 3.6 and 3.1 show that NT is in $S\big(T(a)\big)$ for some integer N prime to p. Since p may be chosen arbitrarily and $S\big(T(a)\big)$ is closed under addition, $S\big(T(a)\big)$ contains all sufficiently large multiples of T.

As we have seen in Section 1, Theorem I follows from Theorems 1.2 and 1.3.

Example 3.1. Let a be the real root of $x^3 - 2x^2 + x - 1 = 0$. In the notation of Lemma 3.3, $f = -5x^2 + 5x - 1$, $g = 10x^2 - 13x + 4 = (5x-4)(2x-1)$, and $h = -5x^3 + 8x^2 - 5x + 1$. Let $\alpha = .27639\ldots$ be the smaller root of f and let $\beta = .36299\ldots$ be the real root of h. We get equidissections of $T(a)$ by taking x and y rational, x in (α,β), y small and nonnegative, and making the construction of Lemma 3.5.

When $x = 1/3$ and $y = 1/15$, up to a scaling factor, the u_i are 1, 3, 5, 1, 0, 5. It follows that 15 is in $S\big(T(a)\big)$. When $x = 2/7$ and $y = 1/70$, up to a scaling factor, the u_i are 1, 21, 20, 1, 7, 20, and so 70 is in $S\big(T(a)\big)$. Therefore all multiples of 5 beginning with 130 lie in $S\big(T(a)\big)$, in agreement with Theorem I. It should not be hard to show that 5 is not in $S\big(T(a)\big)$, but we do not know whether small multiples of 5 such as 10, 20, or 25 are in $S\big(T(a)\big)$.

We can say a bit more about $S\big(T(a)\big)$. In [3], a zig-zag 6-triangle preliminary dissection (see Fig. 1, p. 288) leads to a fifth degree polynomial equation in a (see equation (1), p. 289). When $\sigma_1 = 10/40$, $\sigma_2 = 15/40$, $\sigma_3 = 20/40$, $\sigma_4 = 24/40$, $\sigma_5 = 32/40$, the equation becomes $(3a^2-a+4)(a^3-2a^2+a-1) = 0$. This says 40 is in $S\big(T(a)\big)$. (The six triangles in the preliminary dissection can be dissected into 10, 5, 5, 4, 8, 8 triangles of equal areas.) Hence all multiples of 5 from 70 on lie in $S\big(T(a)\big)$.

Example 3.2. Let a be the real root of $x^3 - 4x^2 + x - 2 = 0$. We check that $4/(1+a)$ is an algebraic integer and $2/(1+a)$ is not. In Lemma 3.3, $f = -8x^2 + 7x - 1$, $g = (4x-2)(4x-3)$, $h = -8x^3 + 12x^2 - 7x + 1$. Let $\alpha = .1798\ldots$ be the smaller root of f and let $\beta = .2051\ldots$ be the real root of h. The construction of Lemma 3.5 produces equidissections of $T(a)$ when x and y are rational, x is in (α, β), and y is small and nonnegative. Taking $x = 7/36$ and $y = 1/18$, up to a scaling factor, the u_i are 6, 98, 49, 14, 1, 84 and so $252 = (63)4$ is in $S\big(T(a)\big)$. When $x = 1/5$ and $y = 0$, up to a scaling factor, the u_i are 5, 33, 16, 0, 1, 25. So $80 = (20)4$ is in $S\big(T(a)\big)$. Hence all sufficiently large multiples of 4 are in $S\big(T(a)\big)$, again in agreement with Theorem I.

References

[1] Encyclopaedia of Mathematics – Stability Criterion. `http://eom.springer.de/S/s086950.htm`.

[2] C. H. Jepsen, Equidissections of trapezoids, Amer. Math. Monthly 103 (1996), 498-500.

[3] E. A. Kasimatis and S. K. Stein, Equidissections of polygons, Discrete Math. 85 (1990), 281-294.

[4] M. Laczkovich and G. Szekeres, Tilings of the square with similar rectangles, Discrete Comput. Geom. 13 (1995), 569-572.

[5] H. Matsumura, Commutative Ring Theory, Cambridge Studies in Advanced Mathematics #8, Cambridge University Press, 1986.

[6] P. Monsky, On dividing a square into triangles, Amer. Math. Monthly 77 (1970), 161-164.

[7] P. Monsky, Calculating a trapezoidal spectrum, Amer. Math. Monthly 103 (1996), 500-501.

[8] S. K. Stein and S. Szabó, Algebra and Tiling: Homomorphisms in the Service of Geometry, Mathematical Association of America, Washington, DC, 1994.